Computable approximations of semicomputable chainable continua

Vedran Čačić, Marko Horvat, Zvonko Iljazović

21/09/21

Goal of this work

Theorem (IP2018)

Let (X, d, α) be a computable metric space and $S \subseteq X$ a semicomputable, decomposable, chainable continuum. Then for every $\varepsilon > 0$ there exist computable points $\hat{a}, \hat{b} \in S$ and a computable subcontinuum $\hat{S}$ of S such that $\hat{S} \approx_{\varepsilon} S$ and $\hat{S}$ is chainable from $\hat{a}$ to $\hat{b}$.

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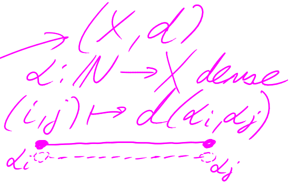

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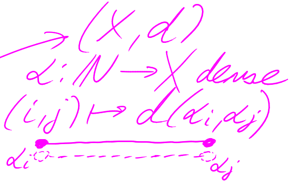

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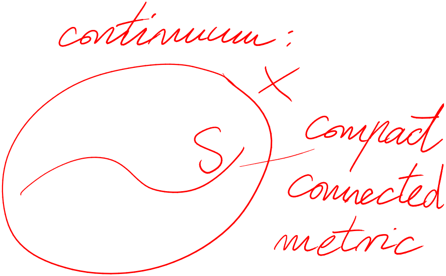

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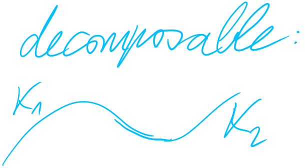

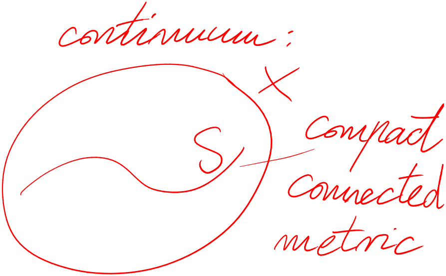

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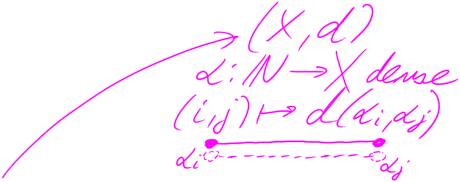

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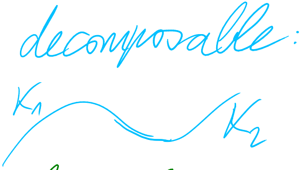

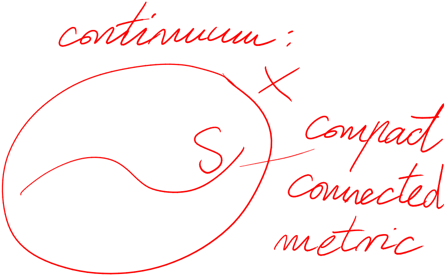

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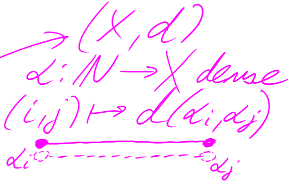

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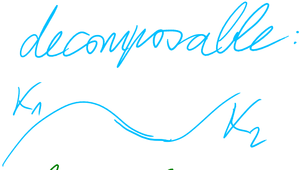

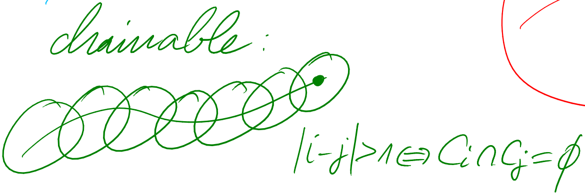

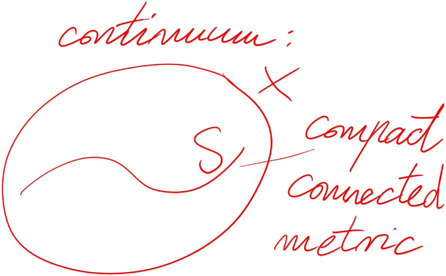

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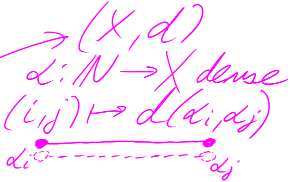

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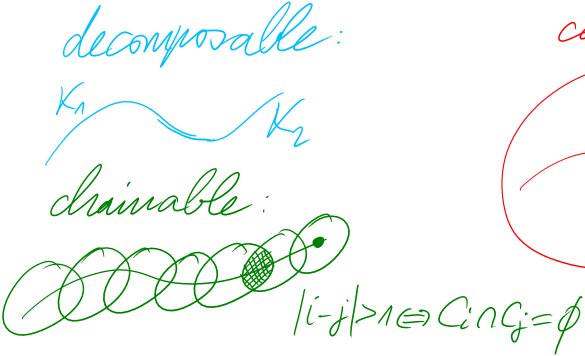

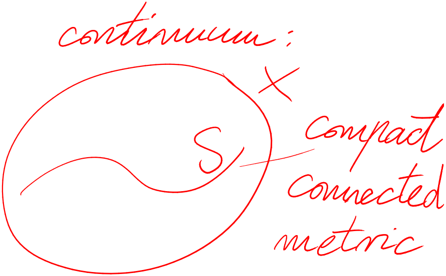

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(X, d)
α : ℕ → X dense
$(i,j) \mapsto d(\alpha_i, \alpha_j)$

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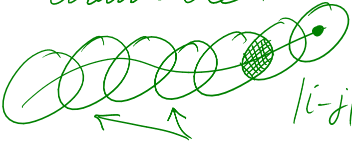
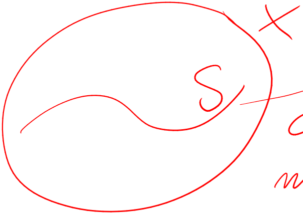
chainable:



$|i-j| > 1 \Leftrightarrow C_i \cap C_j = \emptyset$

continuum:



compact
connected
metric

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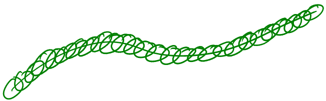

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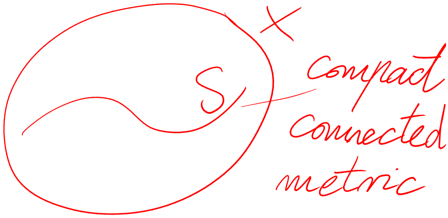

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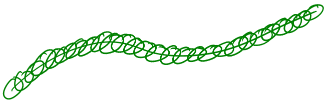
$\hookrightarrow \forall x \in \hat{S}\ \exists y \in S\ d(x,y) < \varepsilon$ (and $\forall x \in S\ \exists y \in \hat{S}$)

decomposable:

chainable:



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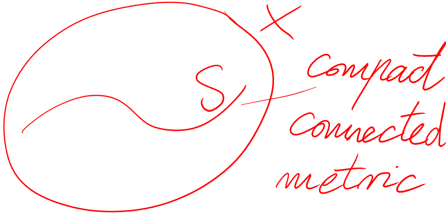

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Theorem (ČHI?)

Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and $S \subseteq X$ a semicomputable, decomposable, chainable Hausdorff continuum. Then for every open cover $\mathcal{U}$ of $(X, \mathcal{T})$ there exist computable points $\hat{a}, \hat{b} \in S$ and a computable subcontinuum $\hat{S}$ of S such that $\hat{S} \approx_{\mathcal{U}} S$ and $\hat{S}$ is chainable from $\hat{a}$ to $\hat{b}$.

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↳ $\forall x \in S \ \exists y \in \hat{S} \ \{x, y\} \subseteq U \in \mathcal{U}$

$\hat{S}$ S

Computable topological space

Definition

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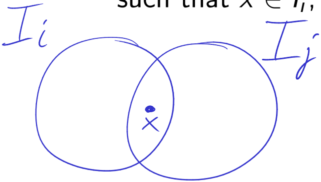

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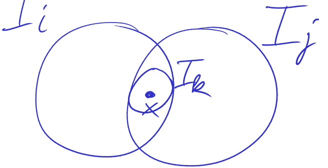

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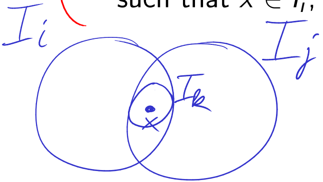

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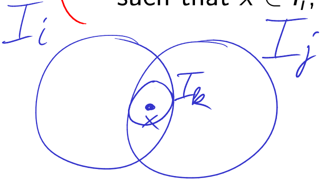

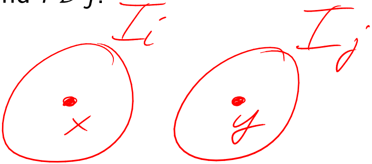

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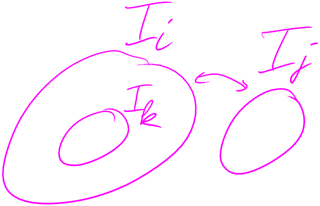

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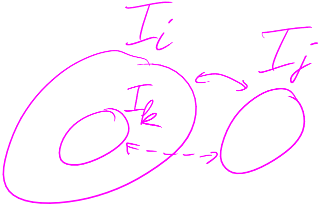

Computable points and sets

Let $[\cdot]$ be a recursive function which enumerates all nonempty finite subsets of $\mathbb{N}$ (every such set is $[j]$ for some $j \in \mathbb{N}$).
We fix $\big(X, \mathcal{T}, (I_i)\big)$ and for $j \in \mathbb{N}$, we define $J_j = \bigcup_{i \in [j]} I_i$.

Definition

Let $x \in X$ and $S \subseteq X$.

- x is a **computable point** in $\big(X, \mathcal{T}, (I_i)\big)$
  if $\{i \in \mathbb{N} : x \in I_i\}$ is c.e.
- S is **computably enumerable** (or **c.e.**) in $\big(X, \mathcal{T}, (I_i)\big)$
  if S is closed in $(X, \mathcal{T})$ and $\{i \in \mathbb{N} : I_i \cap S \neq \emptyset\}$ is c.e.
- S is **semicomputable** in $\big(X, \mathcal{T}, (I_i)\big)$
  if S is compact in $(X, \mathcal{T})$ and $\{j \in \mathbb{N} : S \subseteq J_j\}$ is c.e.
- S is **computable** in $\big(X, \mathcal{T}, (I_i)\big)$
  if S is semicomputable and c.e. in $\big(X, \mathcal{T}, (I_i)\big)$.

Starting with a simpler and more concrete statement

Theorem

Let $(S, \mathcal{T}, (I_i))$ be a semicomputable, decomposable, chainable Hausdorff continuum. Then for every open cover $\mathcal{U}$ of $(S, \mathcal{T})$ there exist computable points $\hat{a}, \hat{b} \in S$ and a computable subcontinuum $\hat{S}$ of S such that $\hat{S} \approx_{\mathcal{U}} S$ and $\hat{S}$ is chainable from $\hat{a}$ to $\hat{b}$.

Theorem

Let $(S, \mathcal{T}, (I_i))$ be a semicomputable, chainable Hausdorff continuum. Let K_1 and K_2 be subcontinua of S such that $S = K_1 \cup K_2$ and let $a \in K_1 \setminus K_2$ and $b \in K_2 \setminus K_1$. Finally, let $\alpha, \beta \in \mathbb{N}$ such that $a \in I_\alpha$ and $b \in I_\beta$. Then there exist computable points $\hat{a}, \hat{b} \in S$ and a computable subcontinuum $\hat{S}$ of S such that $\hat{a} \in I_\alpha$, $\hat{b} \in I_\beta$ and $\hat{S}$ is chainable from $\hat{a}$ to $\hat{b}$.

The theorem and the construction stages

$(S, \mathcal{T}, (I_i))$ semicomp., chainable Hausdorff cont., K_1 and K_2 subc. of S, $S = K_1 \cup K_2$, $a \in K_1 \setminus K_2$, $b \in K_2 \setminus K_1$, $\alpha, \beta \in \mathbb{N}$, $a \in I_\alpha$, $b \in I_\beta$. Then there exist comp. points $\hat{a}, \hat{b} \in S$ and a comp. subc. $\hat{S}$ of S such that $\hat{a} \in I_\alpha$, $\hat{b} \in I_\beta$ and $\hat{S}$ is chainable from $\hat{a}$ to $\hat{b}$.

Construction stages for $\hat{S}, \hat{a}, \hat{b}$.

1. Enlarge K_1, K_2 to $J_{\tilde{a}}, J_{\tilde{b}}$ so that $a \in S \setminus J_{\tilde{b}}$ and $b \in S \setminus J_{\tilde{a}}$
2. The set of chains that cover S so that the first two links are in $J_{\tilde{a}}$, the last two in $J_{\tilde{b}}$, and the rest in $J_{\tilde{a}}$ and $J_{\tilde{b}}$ is c.e.
3. Rec. select a „falling'' sequence $((p'_n, l_n, q'_n))_n$ of these and let $\hat{S} = \bigcap_{n \in \mathbb{N}} (\overline{J_{(l_n)_0}} \cup \ldots \cup \overline{J_{(l_n)_{\overline{l_n}}}})$, $\hat{a} \in \bigcap_{n \in \mathbb{N}} \overline{J_{(l_n)_0}}$, $\hat{b} \in \bigcap_{n \in \mathbb{N}} \overline{J_{(l_n)_{\overline{l_n}}}}$

The theorem and the construction stages

$(S, \mathcal{T}, (I_i))$ semicomp., chainable Hausdorff cont., K_1 and K_2 subc. of S, $S = K_1 \cup K_2$, $a \in K_1 \setminus K_2$, $b \in K_2 \setminus K_1$, $\alpha, \beta \in \mathbb{N}$, $a \in I_\alpha$, $b \in I_\beta$. Then there exist comp. points $\hat{a}, \hat{b} \in S$ and a comp. subc. $\hat{S}$ of S such that $\hat{a} \in I_\alpha$, $\hat{b} \in I_\beta$ and $\hat{S}$ is chainable from $\hat{a}$ to $\hat{b}$.

Construction stages for $\hat{S}, \hat{a}, \hat{b}$.

1. Enlarge K_1, K_2 to $J_{\tilde{a}}, J_{\tilde{b}}$ so that $a \in S \setminus J_{\tilde{b}}$ and $b \in S \setminus J_{\tilde{a}}$
2. The set of chains that cover S so that the first two links are in $J_{\tilde{a}}$, the last two in $J_{\tilde{b}}$, and the rest in $J_{\tilde{a}}$ and $J_{\tilde{b}}$ is c.e.
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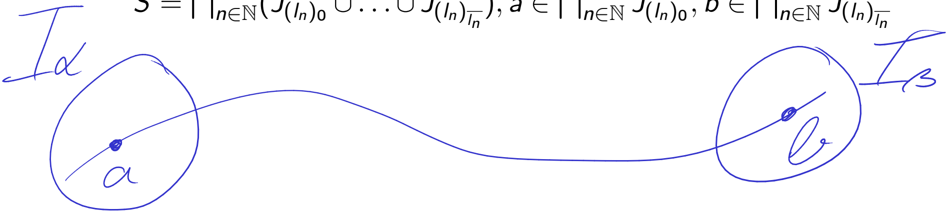

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Construction stages for $\hat{S}, \hat{a}, \hat{b}$.

1. Enlarge K_1, K_2 to $J_{\tilde{a}}, J_{\tilde{b}}$ so that $a \in S \setminus J_{\tilde{b}}$ and $b \in S \setminus J_{\tilde{a}}$
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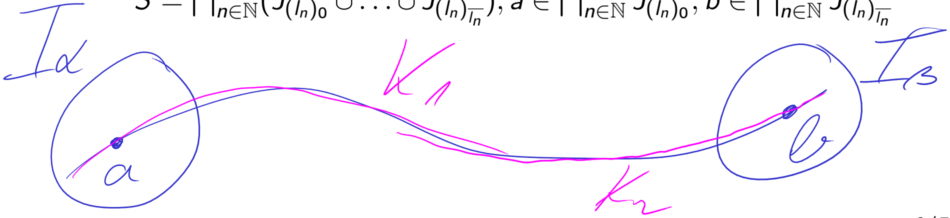

The theorem and the construction stages

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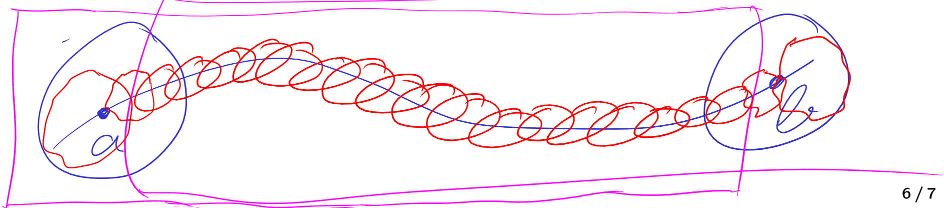

The theorem and the construction stages

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Construction stages for $\hat{S}, \hat{a}, \hat{b}$.

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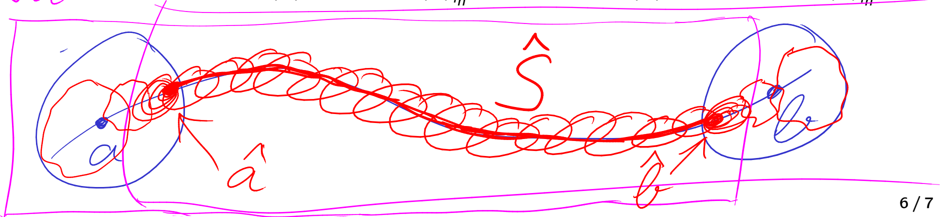

A simple corollary

A topological space A is an **arc** if there exists a homeomorphism $f : [0, 1] \to A$. We say that $f(0)$ and $f(1)$ are the **endpoints** of A.

Corollary

Let $\big(X, \mathcal{T}, (I_i)\big)$ be a computable topological space, S a semicomputable set which is an arc as a subspace of $(X, \mathcal{T})$. Then for all $\alpha, \beta \in \mathbb{N}$ such that I_α and I_β intersect S, there exist different computable points $a \in I_\alpha \cap S$ and $b \in I_\beta \cap S$ such that the subarc of S with endpoints a and b is a computable set in $\big(X, \mathcal{T}, (I_i)\big)$.