n-bisimulation for generalised Veltman semantics

Sebastijan Horvat, Tin Perkov, Mladen Vuković

Faculty of science, University of Zagreb
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Contents

Interpretability logic IL

We will assume that you're familiar with the following concepts:

- interpretability logic $\mathtt{IL}$, Veltman frames and Veltman models
- generalised Veltman frame
- A **generalised Veltman model** is a quadruple $\mathfrak{M} = \Big(W, R, \{S_w \mid w \in W\}, \Vdash\Big)$, where the first three components form a generalised Veltman frame and where V is a valuation mapping propositional variables to subsets of W. The forcing relation $\mathfrak{M}, w \Vdash A$ is defined as in definition of Veltman models with the difference that now

$$\mathfrak{M}, w \Vdash A \rhd B \quad \Leftrightarrow \quad \forall u \Big(wRu \ \&\ u \Vdash A \ \Rightarrow\ \exists V (uS_w V \ \&\ V \Vdash B)\Big).$$

Modal depth

V. Čačić, D. Vrgoč, *A Note on Bisimulation and Modal Equivalence in Provability Logic and Interpretability Logic*, Studia Logica 101(2013), 31–44

Modal depth is a mapping $d : Form_{IL} \to \mathbb{N}$ defined as follows:

- $d(p) = 0$ for all $p \in Prop$,
- $d(\perp) = 0$,
- $d(\neg F) = d(F)$,
- $d(F \vee G) = \max\{d(F), d(G)\}$,
- $d(F \rhd G) = 1 + \max\{d(F), d(G)\}$.

Modally n-equivalent worlds

Let $\mathfrak{M} = (W, R, (S_w)_w, V)$ and $\mathfrak{M}' = (W', R', (S'_{w'})_{w'}, V')$ be two Veltman models, and let $w \in W$ and $w' \in W'$ be worlds in them. Let $n \in \mathbb{N}$.
We say that w and w' are

- **modally equivalent**, and write $\mathfrak{M}, w \equiv \mathfrak{M}', w'$, if for every $\mathtt{IL}$-formula F,

$$\mathfrak{M}, w \Vdash F \text{ iff } \mathfrak{M}', w' \Vdash F,$$

- **modally n-equivalent**, and write $\mathfrak{M}, w \equiv_n \mathfrak{M}', w'$, if for every $\mathtt{IL}$-formula F of modal depth not more than n,

$$\mathfrak{M}, w \Vdash F \text{ iff } \mathfrak{M}', w' \Vdash F,$$

- **propositional equivalent** if they agree on all propositional variables.

n-bisimulations (1/2)

Let $n \in \mathbb{N}$. An ***n*–bisimulation** between two generalised Veltman models $\mathfrak{M} = (W, R, S, \Vdash)$ and $\mathfrak{M}' = (W', R', S', \Vdash')$ is a decreasing sequence of relations

$$Z_n \subseteq Z_{n-1} \subseteq \cdots \subseteq Z_1 \subseteq Z_0 \subseteq W \times W'$$

that possesses the following properties:

(at) for every $(w, w') \in Z_0$, for every $p \in Prop$,

$$w \Vdash p \quad \text{iff} \quad w' \Vdash' p,$$

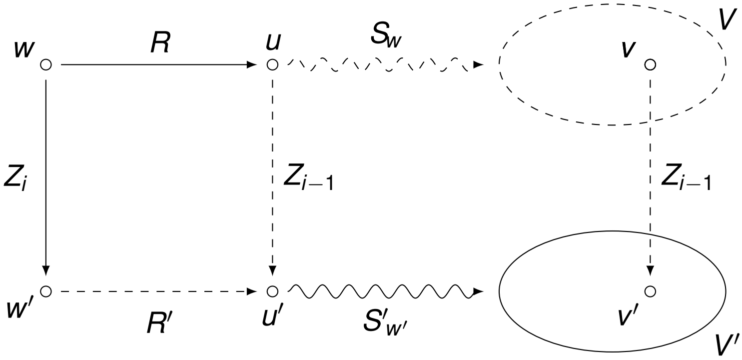
(forth) for every i from 1 to n:
for every $(w, w') \in Z_i$, for every u such that wRu, there exists u' such that $uZ_{i-1}u'$, $w'R'u'$, and for every V' such that $u'S'_{w'}V'$, there exists V such that uS_wV, and for every $v \in V$ there exists $v' \in V'$ such that $vZ_{i-1}v'$,

(back) for every i from 1 to n:
for every $(w, w') \in Z_i$, for every u' such that $w'R'u'$, there exists u such that $uZ_{i-1}u'$, wRu, and for every V such that uS_wV, there exists V' such that $u'S'_{w'}V'$, and for every $v' \in V'$ there exists $v \in V$ such that $vZ_{i-1}v'$.

We say that $w \in W$ and $w' \in W'$ are n**–bisimilar**, and write $\mathfrak{M}, w \leftrightarrows_n \mathfrak{M}', w'$, if there is an n–bisimulation $Z_n \subseteq \cdots \subseteq Z_1 \subseteq Z_0 \subseteq W \times W'$, such that wZ_nw'.

■

The (forth) property



n-bisimilarity implies n-eqivalence

Theorem

Let $\mathfrak{M} = (W, R, S, \Vdash)$ and $\mathfrak{M}' = (W', R', S', \Vdash')$ be two generalised Veltman models. Then for every $n \in \mathbb{N}$,

$$\mathfrak{M}, w \leftrightarrows_n \mathfrak{M}', w' \quad \text{implies} \quad \mathfrak{M}, w \equiv_n \mathfrak{M}', w',$$

for all $w \in W$ and $w' \in W'$.

- easy inductive proof
- we will show that the **converse does not hold**

A method for „lifting” Veltman models to generalised Veltman models

Let $\mathfrak{M} = (W, R, \{S'_w : w \in W\}, \Vdash')$ be a Veltman model.

We define the generalised Veltman model $Gen\,\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$, where for every $w \in W$, $V \subseteq R[w]$ and $v \in R[w]$, we define

$$vS_w V \quad \text{iff} \quad (\exists u \in V)(vS'_w u).$$

The forcing relation $\Vdash$ is defined such that it agrees with $\Vdash'$ on propositional variables, and is extended by definition on complex formulas.

- Similarly one could define the generalised Veltman **frame** $Gen\,\mathfrak{F}$ for Veltman frame $\mathfrak{F}$.

Preservation of modal equivalence

- It is easy to check that $\mathit{Gen}\,\mathfrak{M}$ is a generalised Veltman model.
- By induction on the complexity of a formula we can prove the following equivalence:

Theorem

Let $\mathfrak{M} = (W, R, S', \Vdash')$ be a Veltman model, and let $\mathit{Gen}\,\mathfrak{M} = (W, R, S, \Vdash)$ be a generalised Veltman model. For all formulas F and every $w \in W$ we have

$$\mathfrak{M}, w \Vdash' F \quad \text{iff} \quad \mathit{Gen}\,\mathfrak{M}, w \Vdash F.$$

Bisimulation

An **bisimulation** between two generalised Veltman models $\mathfrak{M} = (W, R, S, \Vdash)$ and $\mathfrak{M}' = (W', R', S', \Vdash')$ is a relation $Z \subseteq W \times W'$ that possesses the following properties:

(gen-at) for every $(w, w') \in Z$, for every $p \in Prop$, $w \Vdash p$ iff $w' \Vdash p$,

(gen-forth) for every $(w, w') \in Z$, for every u such that wRu, there exists u' such that uZu', $w'R'u'$, and for every V' such that $u'S'_{w'}V'$, there exists V such that uS_wV, and for every $v \in V$ there exists $v' \in V'$ such that vZv',

(gen-back) for every $(w, w') \in Z$, for every u' such that $w'R'u'$, there exists u such that uZu', wRu, and for every V such that uS_wV, there exists V' such that $u'S'_{w'}V'$, and for every $v' \in V'$ there exists $v \in V$ such that vZv'.

We say that $w \in W$ and $w' \in W$ are **bisimilar**, and write $\mathfrak{M}, w \leftrightarrows \mathfrak{M}', w'$, if there is a bisimulation $Z \subseteq W \times W'$ such that wZw'.

Theorem

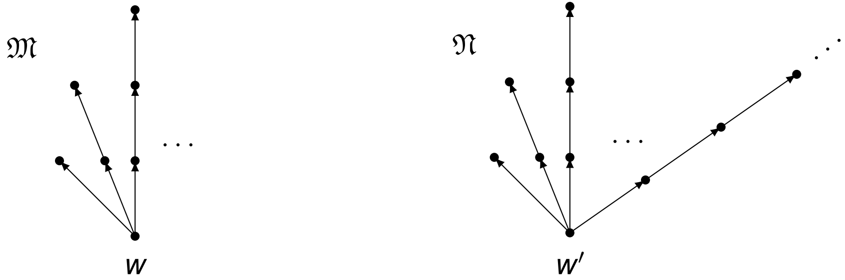
Let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ and $\mathfrak{M}' = (W', R', \{S'_w : w \in W\}, \Vdash')$ be two Veltman models and $w_0 \in W$, $w'_0 \in W'$ be worlds in them, respectively. Then

$$Gen\,\mathfrak{M}, w_0 \leftrightarroweq Gen\,\mathfrak{M}', w'_0 \quad \text{iff} \quad \mathfrak{M}, w_0 \leftrightarroweq \mathfrak{M}', w'_0.$$

- Preservation of n-bisimulation can be proven analogously.

Modal equivalence does not imply bisimilarity

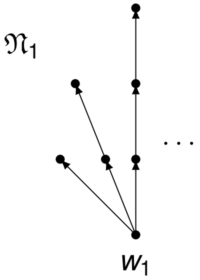
Kripke models:

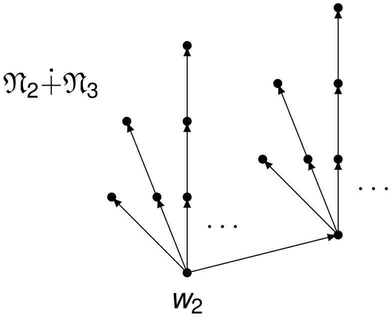


Slika: w and w' are modally equivalent but not bisimilar

Modal equivalence does not imply bisimilarity

Veltman models:





Slika: w and w' are modally equivalent but not bisimilar

Modal equivalence does not imply bisimilarity

A method for obtaining Veltman models from GL-models

- Let $\mathfrak{N} = (W, R, V)$ be a **GL**-model. For every $w \in W$ we define

$$uS_w v \quad \text{iff} \quad wRu\underline{R}v,$$

where we denote the reflexive closure of R with $\underline{R}$. We denote $(W, R, \{S_w : w \in W\}, V)$ by $\mathit{Vel}\ \mathfrak{N}$.

Theorem

The worlds w_1 and w_2, in Veltman models $\mathfrak{M}_1 \equiv \mathit{Vel}\ \mathfrak{N}_1$ and $\mathfrak{M}_2 \equiv \mathit{Vel}\ (\mathfrak{N}_1 \dot{+} \mathfrak{N}_2)$, are modally equivalent but not bisimilar.

Bisimulation is strictly stronger than modal equivalence

Theorem

The worlds w_1 and w_2 (from prevoius slide), in **generalised Veltman models** *Gen* $\mathfrak{M}_1$ and *Gen* $\mathfrak{M}_2$, are modally equivalent but not bisimilar.

References:

- the notion of game for **Kripke models**:

> V. Goranko, M. Otto, *Model theory for modal logic*, In: P. Blackburn P., J. van Benthem, F. Wolter (eds.) Handbook of Modal Logic, pp.249-329, Elsevier, Amsterdam (2006)

- the notion of game for **Veltman models**:

> V. Čačić, D. Vrgoč, *A Note on Bisimulation and Modal Equivalence in Provability Logic and Interpretability Logic*, Studia Logica 101(2013), 31–44

Bisimulation game for generalised Veltman semantics

- Let $\mathfrak{M}_i = \left(W_i, R_i, \{S_w^{(i)} : w \in W_i\}, \Vdash_i\right)$, $i \in \{0, 1\}$, be two generalised Veltman models.
- The **bisimulation game** is played by two players, *challenger* and *defender* who move from one configuration to another in a series of consecutive rounds.
- A configuration is a 4-tuple $(\mathfrak{M}_0, w_0, \mathfrak{M}_1, w_1)$, where $w_0 \in W_0$ and $w_1 \in W_1$. Each round starts with some configuration $(\mathfrak{M}_0, w_0, \mathfrak{M}_1, w_1)$. At the beginning of each round it is checked that $\mathfrak{M}_0, w_0 \equiv_0 \mathfrak{M}_1, w_1$. If that check fails, challenger wins.

Bisimulation game for generalised Veltman semantics

A single round, starting with the configuration $(\mathfrak{M}_0, w_0, \mathfrak{M}_1, w_1)$, is played as follows:

1. Challenger chooses $i \in \{0, 1\}$, index of one generalised Veltman model. We denote $j := 1 - i$, the index of another model.
2. Challenger picks $u_i \in W_i$ such that $w_i R_i u_i$. If there are no such worlds, the defender wins and game is over.
3. Defender picks $u_j \in W_j$ such that $w_j R_j u_j$. If there are no such worlds, the challenger wins and game is over.
4. Challenger picks $V_j \subseteq W_j$ such that $u_j S^{(j)}_{w_j} V_j$.
5. Defender picks $V_i \subseteq W_i$ such that $u_i S^{(i)}_{w_i} V_i$.

Bisimulation game for generalised Veltman semantics

The next round is then played from the configuration $(\mathfrak{M}_0, w, \mathfrak{M}_1, w')$ where

(i) challenger chooses u_i or $v_i \in V_i$.

(ii) In case the challenger has chosen u_i, the next round is played from configuration $(\mathfrak{M}_0, u_0, \mathfrak{M}_1, u_1)$. In case the challenger has chosen $v_i \in V_i$, then **the defender chooses** $v_j \in V_j$, and the next round is played from configuration $(\mathfrak{M}_0, v_0, \mathfrak{M}_1, v_1)$.

- notice the difference between definition of bisimulation games for Veltman models and generalised Veltman models

Remarks

- for every $n \in \mathbb{N}$, one can define an n-game:

> An ***n*-game** is a bisimulation game with the following rule added: if n rounds have been played, and challenger hasn't won, then defender wins and the game ends.

- An 0-game is a bisimulation game without any round played. In an 0-game that starts from the configuration $(\mathfrak{M}_0, w_0, \mathfrak{M}_1, w_1)$, defender has a winning strategy if $\mathfrak{M}_0, w_0 \equiv_0 \mathfrak{M}_1, w_1$.
- Note that steps 4. and 5. can always be played. For instance, because relation $S_{w_j}^{(j)}$ is quasi-reflexive, challenger can always pick $V_j := \{u_j\}$.

Remarks

- it can easily be proven:

Proposition

Every bisimulation game ends, i.e., there are no infinite games.

- Some results on bisimulation games can be found in:
  - V. Čačić, D. Vrgoč, *A Note on Bisimulation and Modal Equivalence in Provability Logic and Interpretability Logic*, Studia Logica 101(2013), 31–44
  - T. Perkov, M. Vuković, *A bisimulation characterization for interpretability logic*, Logic Journal of the IGLP 22(2014), 872–879

Proposition

Let $\mathfrak{M}_0$ and $\mathfrak{M}_1$ be two generalised Veltman models and $w_0 \in W_0$, $w_1 \in W_1$ be worlds in them, respectively. For every $n \in \mathbb{N}$, defender has a winning strategy in an n-game with a starting configuration $(\mathfrak{M}_0, w_0, \mathfrak{M}_1, w_1)$ iff $\mathfrak{M}_0, w_0 \leftrightarroweq_n \mathfrak{M}_1, w_1$.

The End.

Questions?