KNEALE'S DEVELOPEMENTS AS BETH'S TABLEAUS

Z. ŠIKIĆ

DEDICATED TO THE LATE KOSTA DOŠEN WHO ASKED ME TO WRITE IT

SINTACTIC TRADITION

AXIOMATIZATION:

(F1) $A \to (B \to A)$

(F2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(F3) $(A \to (B \to C)) \to (B \to (A \to C))$

(Cp) $(A \to B) \to (-B \to -A)$

(DNI) $A \to --A$ (DNE) $--A \to A$

(F4) $\forall x F \to F(a/x)$

(F5) $\forall x (A \to F) \to (A \to \forall x F)$ $a \notin F$

(MP) $\dfrac{A \quad A \to B}{B}$ (G) $\dfrac{F(a/x)}{\forall x F}$

HARD TO HANDLE WITHOUT DERIVED RULES!!

AXIOMATIZATION ≡ DERIVED RULES (NATURAL DEDUCTIONS)

(I→)
$$\dfrac{\dfrac{}{A}^{*} \\ \vdots \\ B}{A \to B}^{*}$$

(E→)
$$\dfrac{A \quad A \to B}{B}$$

(I⊥) (E¬)
$$\dfrac{A \quad \neg A}{\bot}$$

(E⊥) (I¬)
$$\dfrac{\dfrac{}{A}^{*} \\ \vdots \\ \bot}{\neg A}^{*}$$

(I∀)
$$\dfrac{F(a/x)}{\forall x F} \quad \cancel{a}$$

(E∀)
$$\dfrac{\forall x F}{F(a/x)}$$

(E¬¬)
$$\dfrac{\neg\neg A}{A}$$

ŁUKASIEWICZ:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A\to(B\to C)}^{***} \quad \dfrac{}{A}^{*}}{B\to C} \quad \dfrac{}{B}^{**}}{C}}{A\to C}^{*}}{B\to(A\to C)}^{**}}{(A\to(B\to C))\to(B\to(A\to C))}^{***} \quad (F3)$$

DOKAZ TEOREMA DEDUKCIJE $(I\to)$:

$\vdots$

C Ax ili Pret

$C\to(A\to C)$ F1

$\boxed{A\to C}$ MP

$$\dfrac{\dfrac{(F_1)^{ABC}_{AAA} \qquad (F_2)^{ABC}_{A\,A\to A\,A}}{(F_1)^{ABC}_{AAA}\to(A\to A)} \qquad (F_1)^{ABC}_{AAA}}{\boxed{A\to A}}$$

$\vdots$

$A\to(B\to C)$ Pret. Ind

$(A\to(B\to C))\to((A\to B)\to(A\to C))$ F2

$(A\to B)\to(A\to C)$ MP

$\vdots$

$A\to B$ Pret Ind

$\boxed{A\to C}$ MP

Iz $(I\to)$ i $(E\to)$ slijede (F_1) (F_2):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A}^{(1)} \quad \overline{A\to(B\to C)}^{(3)}}{B\to C} \qquad \dfrac{\overline{A}^{(1)} \quad \overline{A\to B}^{(2)}}{B}}{C}}{A\to C}(1)}{(A\to B)\to(A\to C)}(2)}{\boxed{(A\to(B\to C))\to((A\to B)\to(A\to C))}}(3)$$

$$\dfrac{\dfrac{\overline{A}^{(1)}}{B\to A}*}{\boxed{A\to(B\to A)}}(1)$$

(Cp) i (DNI) SLIJEDE IZ (IE→, ⊥, ¬):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A\to B]^{(3)} \qquad [A]^{(1)}}{B} \qquad [\neg B]^{(2)}}{\bot}}{\neg A}\,(1)}{\neg B\to\neg A}\,(2)}{(A\to B)\to(\neg B\to\neg A)}\,(3)$$

$$\dfrac{\dfrac{\dfrac{[A]^{(2)} \qquad [\neg A]^{(1)}}{\bot}}{\neg\neg A}\,(1)}{A\to\neg\neg A}\,(2)$$

OBRATI SLIJEDE UZ $\bot := \neg(C\to C)$:

$$\dfrac{\overline{C\to C} \qquad \dfrac{\dfrac{\begin{matrix}[A]^{(1)}\\ \vdots \\ \neg(C\to C)\end{matrix}}{A\to\neg(C\to C)}\,(1)}{(C\to C)\to\neg A}\,(Cp)}{\neg A}$$

$$\dfrac{\dfrac{\dfrac{A}{\neg B\to A}}{\neg A\to B}\,(Cp) \qquad \neg A}{B}$$

MOŽE BITI I $\bot$

(ED¬) DAJE I TAJ JAČI

(EFQ) VIDI DALJE

WHEN YOU INTRODUCE $\wedge, \vee, \exists$: $\boxed{\text{GND}}$

(I∧) $$\frac{A \quad B}{A \wedge B}$$

(E∧) $$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

(I∨) $$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B}$$

(E∨) $$\frac{A \vee B \quad \begin{array}{c} \overline{A}^{\,*} \\ \vdots \\ D \end{array} \quad \begin{array}{c} \overline{B}^{\,*} \\ \vdots \\ D \end{array}}{D}\,*$$

(I→) $$\frac{\begin{array}{c} \overline{A}^{\,*} \\ \vdots \\ B \end{array}}{A \to B}\,*$$

(E→) $$\frac{A \quad A \to B}{B}$$

(I-)
(E⊥) $$\frac{\begin{array}{c} \overline{A}^{\,*} \\ \vdots \\ \perp \end{array}}{-A}\,*$$

(E-)
(I⊥) $$\frac{A \quad -A}{\perp}$$

(I∀) $$\frac{F(a/x)}{\forall x F}\ \not a$$

(E∀) $$\frac{\forall x F}{F(a/x)}$$

(I∃) $$\frac{F(a/x)}{\exists x F}$$

(E∃) $$\frac{\exists x F \quad \begin{array}{c} F(a/x) \\ \vdots \\ D \end{array}}{D}\ \not a$$

(E¬-) $$\frac{--A}{A}$$

$$\left(\frac{\perp}{A} \right)$$

$$\boxed{(ED-) \equiv (TND) + (EFQ)}$$

$$\dfrac{\dfrac{}{A \vee \neg A}\,(TND) \qquad \dfrac{}{A}^{\,*} \qquad \dfrac{\dfrac{\dfrac{}{\neg A}^{\,*} \quad \boxed{\neg\neg A}}{\bot}}{A}\,(EFQ)}{\boxed{A}}\,{*}$$

$$\dfrac{\dfrac{\boxed{\bot}}{\neg\neg A}}{\boxed{A}}\,(ED-)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A}^{\,*}}{A \vee \neg A} \qquad \dfrac{}{\neg(A \vee \neg A)}^{\,**}}{\bot}}{\neg A}\,{*}}{A \vee \neg A} \qquad \dfrac{}{\neg(A \vee \neg A)}^{\,**}}{\bot}}{\neg\neg(A \vee \neg A)}\,{**}}{\boxed{A \vee \neg A}}\,(ED-)$$

SEQUENT FORMAT OF GND:

$$\frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B} \qquad \left(\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \right) \equiv \frac{A, \Gamma \vdash D}{A \wedge B, \Gamma \vdash D}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \left(\frac{\Gamma_1, A \vdash D \quad \Gamma_2, B \vdash D \quad \Gamma_3 \vdash A \vee B}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash D} \right) \equiv \frac{\Gamma_1, A \vdash D \quad \Gamma_2, B \vdash D}{\Gamma_1, \Gamma_2, A \vee B \vdash D}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \qquad \left(\frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash A \to B}{\Gamma_1, \Gamma_2 \vdash B} \right) \equiv \frac{\Gamma_1 \vdash A \quad B, \Gamma_2 \vdash D}{\Gamma_1, \Gamma_2, A \to B \vdash D}$$

$$\frac{A, \Gamma \vdash \bot}{\Gamma \vdash -A} \qquad \left(\frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash -A}{\Gamma_1, \Gamma_2 \vdash \bot} \right) \equiv \frac{\Gamma \vdash A}{\Gamma, -A \vdash \bot}$$

$$\frac{\Gamma \vdash F(a/x)}{\Gamma \vdash \forall x F} \; a \notin \Gamma \qquad \left(\frac{\Gamma \vdash \forall x F}{\Gamma \vdash F(a/x)} \right) \equiv \frac{F(a/x), \Gamma \vdash D}{\forall x F, \Gamma \vdash D}$$

$$\frac{\Gamma \vdash F(a/x)}{\Gamma \vdash \exists x F} \qquad \left(\frac{\Gamma_1 \vdash \exists x F \quad \Gamma_2, F(a/x) \vdash D}{\Gamma_1, \Gamma_2 \vdash D} \right) \equiv \frac{\Gamma, F(a/x) \vdash D}{\Gamma, \exists x F \vdash D} \; a \notin \Gamma$$

$$\dfrac{\dfrac{\dfrac{}{A\wedge B\vdash A\wedge B}}{A\wedge B\vdash A}\;E \qquad \Gamma, A\vdash D}{A\wedge B, \Gamma\vdash D}$$

$$\dfrac{\Gamma\vdash A\wedge B \qquad \dfrac{\dfrac{}{A\vdash A}}{A\wedge B\vdash A}\;I}{\Gamma\vdash A}$$

$$\dfrac{\dfrac{}{A\vee B\vdash A\vee B} \qquad A,\Gamma_1\vdash D \qquad B,\Gamma_2\vdash D}{A\vee B,\Gamma_1,\Gamma_2\vdash D}$$

$$\dfrac{\Gamma_1\vdash A\vee B \qquad \dfrac{A,\Gamma_2\vdash D \qquad B,\Gamma_3\vdash D}{A\vee B,\Gamma_2,\Gamma_3\vdash D}\;I}{\Gamma_1,\Gamma_2,\Gamma_3\vdash D}$$

MULTIPLE (EV)

$$\dfrac{\Gamma\vdash A\vee B}{\Gamma\vdash A, B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\vee B\vdash A\vee B}}{A\vee B\vdash A, B}\;E \qquad A,\Gamma_1\vdash D}{A\vee B,\Gamma_1\vdash B, D} \qquad B,\Gamma_2\vdash D}{A\vee B,\Gamma_1,\Gamma_2\vdash D}$$

$$\dfrac{\Gamma\vdash A\vee B \qquad \dfrac{\dfrac{}{A\vdash A, B} \qquad \dfrac{}{B\vdash A, B}}{A\vee B\vdash A, B}\;I}{\Gamma\vdash A, B}$$

$$\dfrac{\dfrac{\Gamma_1\vdash A \qquad \dfrac{\dfrac{}{A\vdash A} \qquad \dfrac{}{A\to B\vdash A\to B}}{A, A\to B\vdash B}\;E}{\Gamma_1, A\to B\vdash B} \qquad B,\Gamma_2\vdash D}{\Gamma_1,\Gamma_2, A\to B\vdash D}$$

$$\dfrac{\Gamma_2\vdash A\to B \qquad \dfrac{\Gamma_1\vdash A \qquad \dfrac{\dfrac{}{A\to A} \qquad \dfrac{}{B\to B}}{A, A\to B\vdash B}\;I}{\Gamma_1, A\to B\vdash B}}{\Gamma_1,\Gamma_2\vdash B}$$

$$\dfrac{\Gamma \vdash A \qquad \dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{-A \vdash -A}}{A, -A \vdash \bot}\,\text{(E)}}{\Gamma, -A \vdash \bot}$$

$$\dfrac{\Gamma_2 \vdash -A \qquad \dfrac{\Gamma_1 \vdash A \qquad \dfrac{\dfrac{}{A \vdash A}}{A, -A \vdash \bot}\,(1)}{\Gamma_1, -A \vdash \bot}}{\Gamma_1, \Gamma_2 \vdash \bot}$$

$$\dfrac{\text{(E)}\,\dfrac{\dfrac{}{\forall x F \vdash \forall x F}}{\forall x F \vdash F(a/x)} \qquad F(a/x), \Gamma \vdash D}{\forall x F, \Gamma \vdash D}$$

$$\dfrac{\Gamma \vdash \forall x F \qquad (1)\,\dfrac{\dfrac{}{F(a/x) \vdash F(a/x)}}{\forall x F \vdash F(a/x)}}{\Gamma \vdash F(a/x)}$$

$$\dfrac{\text{(E)}\,\dfrac{\dfrac{}{\exists x F \vdash \exists x F}}{\exists x F \vdash F(a/x)} \qquad F(a/x), \Gamma \vdash D}{\exists x F, \Gamma \vdash D}$$

$\emptyset$

$$\dfrac{\Gamma_1 \vdash \exists x F \qquad (1)\,\dfrac{F(a/x), \Gamma_2 \vdash D}{\exists x F, \Gamma_2 \vdash D}}{\Gamma_1, \Gamma_2 \vdash D}$$

(1) $\bot = \emptyset$ $D = \Delta$ SINGLETON OR $\emptyset$

(2) MULTIPLE SYSTEM (cf. $\vee$)

CONVENIENT FOR CLASSICAL LOGIC:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\vdash A, \overline{A}}}{\overline{\overline{A}} \vdash A}$$

SEMANTIC TRADITION

MEANING OF LOGICAL CONSTANTS:

BETH FORMAT | "GENTZEN" FORMAT

$$\frac{A \to B \quad T}{A \perp \qquad B \; T} \equiv \frac{A \to B \vdash}{\vdash A \qquad B \vdash}$$

$$\begin{array}{l} A \to B \quad \perp \\ \quad A \quad T \\ \quad B \quad \perp \end{array} \equiv \frac{\vdash A \to B}{A \vdash B}$$

$$\begin{array}{l} \exists x F \quad T \\ F(a/x) \quad T \end{array} \equiv \frac{\exists x F \vdash}{F(a/x) \vdash} \quad \text{NEW}$$

$$\begin{array}{l} \exists x F \quad \perp \quad a \\ F(a/x) \quad \perp \end{array} \equiv \frac{\vdash \exists x F}{\vdash F(a/x), \exists x F}$$

ETC.

BETH TESTING

$1 \quad \exists x(Fx \wedge Gx) \quad T$

$2 \quad \exists x Fx \wedge \exists x Gx \quad \bot$

$3_1 \quad Fa \wedge Ga \quad T$

$4_3 \quad Fa \quad T$

$5_3 \quad Ga \quad T$

$6_2 \quad \exists x Fx \ \bot a \qquad \exists x Gx \ \bot a$

$7_6 \quad Fa \ \bot \qquad Ga \ \bot$

GENTZEN TESTING

$1,2 \quad \boxed{\exists x(Fx \wedge Gx) \nvdash \exists x Fx \wedge \exists x Gx}$

$3_1 \quad Fa \wedge Ga \nvdash \exists x Fx \wedge \exists x Gx$

$4,5_3 \quad Fa, Ga \nvdash \exists x Fx \wedge \exists x Gx$

$6_2 \quad Fa, Ga \nvdash \exists x Fx \qquad Fa, Ga \nvdash \exists x Gx$

$7_6 \quad Fa, Ga \nvdash Fa, \exists x Fx \qquad Fa, Ga \nvdash Ga, \exists x Gx$

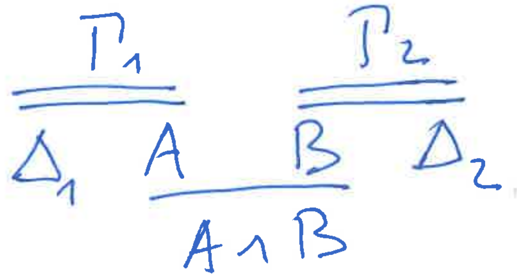
FROM THE BOTTOM UP:
MULTIPLE G. SEQUENT SYST.
WITHOUT CUT

WHERE TO PLACE KNEALE'S SYSTEM OF MULTIPLE NATURAL DEDUCTIONS?

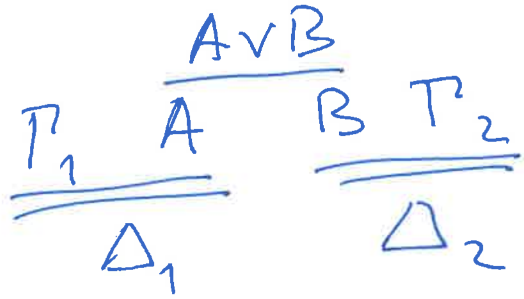
KNEALE INTRODUCES IT AS A GENERALIZATION AND SIMPLIFICATION OF THE GENTZEN'S SINGULAR SYSTEM OF NATURAL DEDUCTIONS IN THE SINTACTIC TRADITION.

WE SEE IT IN THE SEMANTIC TRADITION AS ANOTHER FORMAT OF BETH'S SYSTEM!

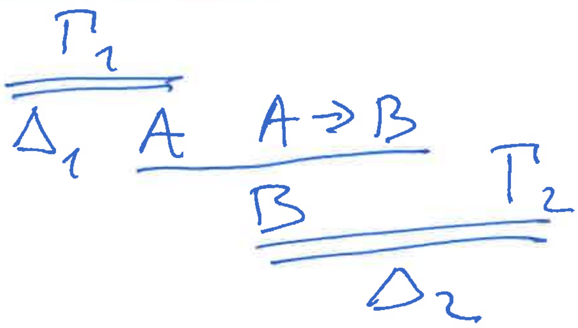
$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \wedge B}$$



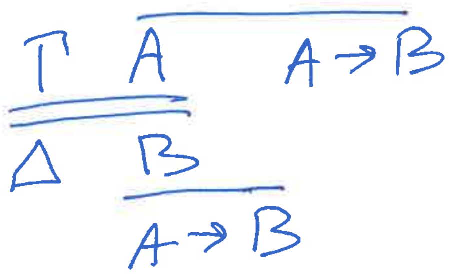
$$\frac{\Gamma_1, A \vdash \Delta_1 \qquad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \vdash \Delta_1, \Delta_2}$$



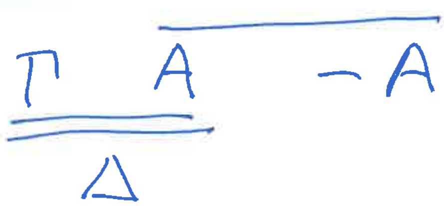
$$\frac{\Gamma_1 \vdash \Delta_1, A \qquad B, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \to B \vdash \Delta_1, \Delta_2}$$



$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}$$



$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}$$



ETC.

$$\frac{\Gamma, F(a/x) \vdash \Delta}{\Gamma, \forall x F \vdash \Delta} \qquad \begin{array}{cc} \Gamma & \dfrac{\forall x F}{F(a/x)} \\ \hline\hline \multicolumn{2}{c}{\Delta} \end{array}$$

$$\cancel{a} \quad \frac{\Gamma \vdash F(a/x), \Delta}{\Gamma \vdash \forall x F, \Delta} \qquad \begin{array}{cc} \multicolumn{2}{c}{\Gamma} \\ \hline\hline \Delta & \dfrac{F(a/x)}{\forall x F} \end{array}$$

$$\cancel{a} \quad \frac{\Gamma, F(a/x) \vdash \Delta}{\Gamma, \exists x F \vdash \Delta} \qquad \begin{array}{cc} \Gamma & \dfrac{\exists x F}{F(a/x)} \\ \hline\hline \multicolumn{2}{c}{\Delta} \end{array}$$

$$\frac{\Gamma \vdash F(a/x), \Delta}{\Gamma \vdash \exists x F, \Delta} \qquad \begin{array}{cc} \multicolumn{2}{c}{\Gamma} \\ \hline\hline \Delta & \dfrac{F(a/x)}{\exists x F} \end{array}$$

RULES $\cancel{a}$ ARE GLOBAL SO THERE IS NO SIMPLIFICATION TO LOCAL RULES (AS KNEALE HAD HOPED)

(W) KNEALE'S SYSTEM:

(I∧) $\dfrac{A \quad B}{A \wedge B}$

(E∧) $\dfrac{A \wedge B}{A} \qquad \dfrac{A \wedge B}{B}$

(I∨) $\dfrac{A}{A \vee B} \qquad \dfrac{B}{A \vee B}$

(E∨) $\dfrac{A \vee B}{A \quad B}$

(I→) $\dfrac{B}{A \to B} \qquad \dfrac{\quad}{A \quad A \to B}$

(E→) $\dfrac{A \quad A \to B}{B}$

(I−) $\dfrac{\quad}{A \quad -A}$

(E−) $\dfrac{A \quad -A}{\quad}$

(I∀)* $\dfrac{F(a/x)}{\forall x F} \quad \cancel{a}$

(E∀) $\dfrac{\forall x F}{F(a/x)}$

(I∃) $\dfrac{F(a/x)}{\exists x F}$

(E∃)* $\dfrac{\exists x F}{F(a/x)} \quad \cancel{a}$

($\overline{W}$)* $\begin{array}{c} \Gamma \quad A \quad \overline{A} \\ \hline\hline \Delta \end{array}$

($\underline{W}$)* $\begin{array}{c} \Gamma \\ \hline\hline \Delta \quad A \quad \underline{A} \end{array}$

* RULES ARE GLOBAL
(NO SIMPLIFICATION)

(W) RULES ARE NECESSARY FOR (E.G.)

$$
\begin{array}{c}
P\vee(Q\wedge R) \\ \hline
\begin{array}{cc}
\begin{array}{c} P \\ \hline P\vee Q \\ \hline {}^{*} \end{array}
&
\begin{array}{c} Q\wedge R \\ \hline Q \\ \hline P\vee Q \end{array}
\end{array}
\end{array}
\qquad\qquad
\begin{array}{c}
P\vee(Q\wedge R) \\ \hline
\begin{array}{cc}
\begin{array}{c} Q\wedge R \\ \hline R \\ \hline P\vee R \end{array}
&
\begin{array}{c} P \\ \hline P\vee R \\ \hline {}^{\times} \end{array}
\end{array}
\end{array}
$$

$$\dfrac{P\vee Q \qquad P\vee R}{(P\vee Q)\wedge(P\vee R)}$$

(W) (W)

$\cancel{\alpha}$ RULES ARE NECESSARY TO NOT ALLOW (E.G.)

$$
\begin{array}{c}
\hline
\begin{array}{cc}
\dfrac{Fa}{\forall x Fx} & \dfrac{-Fa}{\forall x -Fx}
\end{array}
\end{array}
$$

WHICHEVER IS FIRST
IS NOT ALLOWED

EACH GENTZEN'S MS-DEDUCTION GENERATES A (W)KNEALE'S DEDUCTION.

HENCE, (W)KNEALE'S DEDUCTIONS ARE JUST ANOTHER FORMAT OF THE GENTZEN'S MS-DEDUCTIONS.

GENTZEN'S MS-DEDUCTIONS ARE JUST ANOTHER FORMAT OF BETH'S DEDUCTIONS.

HENCE, (W)KNEALE'S DEDUCTIONS ARE JUST ANOTHER FORMAT OF THE BETH'S DEDUCTIONS.