Computable sequences and isometries [1]

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Computability on a metric space

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- a compact set K is **computable** in (X, d, α) if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $K \approx_{2^{-k}} \Lambda_{f(k)}$, for each $k \in \mathbb{N}$.

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Are those notions defined by the metric space itself?

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Under which circumstances are all effective separating sequences equivalent?

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$([\mathbf{0}, \mathbf{1}], \boldsymbol{d}, \boldsymbol{q})$

- if β is an effective separating sequence, 0 is computable in $([0,1], d, \beta)$
- $i \mapsto d(0, \beta_i)$ is computable, so $\beta : \mathbb{N} \to \mathbb{R}$ is computable
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$(\boldsymbol{S^1},\boldsymbol{d},\boldsymbol{\alpha})$

- if x is a computable point and y a non-computable point, there exists a rotation f such that $f(x)=y$
- $f\circ\alpha$ is an effective separating sequence and y is computable in $(S^1,d,f\circ\alpha)$
- $f\circ\alpha \not\sim \alpha$

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(X, d, α) is **an effectively (or computably) compact computable metric space** if (X, d) is complete and there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $X = \bigcup_{i=0}^{f(k)} B(\alpha_i, 2^{-k})$, for each $k \in \mathbb{N}$.

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Theorem (Iljazović, 2010)

Let (X, d, α) be an effectively compact computable metric space such that there exist only finitely many isometries of the metric space (X, d). If β is an effective separating sequence in (X, d), then $\beta \sim \alpha$.

New result

Theorem

Let (X, d, α) be an effectively compact metric space and K a computable compact set in (X, d, α) such that there are only finitely many isometries $f : X \to X$ such that $f(K) \subseteq K$. If β is an effective separating sequence in (X, d) such that K is computable in (X, d, β), then $\alpha \sim \beta$.

Consequence:

Proposition

Assume that (X, d, α) *is an effectively compact computable metric space and* $x_0, \ldots, x_n$ *computable points in* (X, d, α) *such that there are only finitely many isometries* $f : X \to X$ *such that* $f(x_i) = x_i$*,* $i = 0, \ldots, n$*. If* β *is an effective separating sequence such that* $x_0, \ldots, x_n$ *are computable points in* (X, d, β)*, then* $\alpha \sim \beta$*.*

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A metric space is **computably categorical** if every two effective separating sequences are equivalent up to an isometry.

Two examples

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$(\mathbb{R}^n, d, q)$

- if α is an effective separating sequence, then exists an isometry f such that $f \circ \alpha$ is a computable sequence
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$([0, \gamma], d, \alpha)$, γ **left computable, not computable**

- β an effective separating sequence such that $\frac{\gamma}{2}$ is computable in $([0, \gamma], d, \beta)$
- $\frac{\gamma}{2}$ is a fixed point of each isometry and is not computable in $([0, \gamma], d, \alpha)$
- $\alpha \not\sim_{\text{iso}} \beta$

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Proposition $\Rightarrow\ \beta \sim f \circ \alpha\ \Rightarrow\ \beta \sim_{\text{iso}} \alpha$

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(focus on spaces with infinitely many isometries)

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$\tau_1, \tau_2 : \mathbb{N} \to \mathbb{N}$ computable, $\{(\tau_1(i), \tau_2(i)) \mid i \in \mathbb{N}\} = \mathbb{N}^2$
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- If $\{i \in \mathbb{N} \mid S \cap I_i \neq \emptyset\}$ is c.e., we say that S is **computably enumerable** in (X, d, α).
- If there is a c.e. set Ω in $\mathbb{N}$ such that

$$X \setminus S = \bigcup_{i \in \Omega} I_i,$$

we say that S is **co-computably enumerable** in (X, d, α).

Orbits of computable points

Theorem

If (X, d, α) *is an effectively compact computable metric space and* x_0 *a computable point in this space, then*

$$\mathrm{Orb}(x_0) = \{f(x_0) \mid f \in \mathrm{Iso}(X, d)\}$$

is a co-c.e. set.

Computable categoricity of unions of concentric spheres

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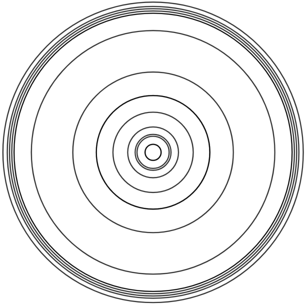

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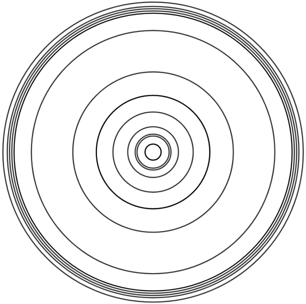

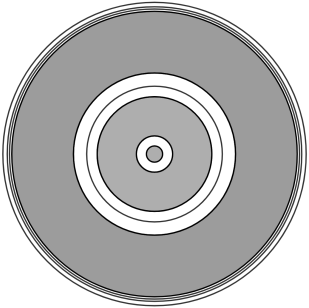

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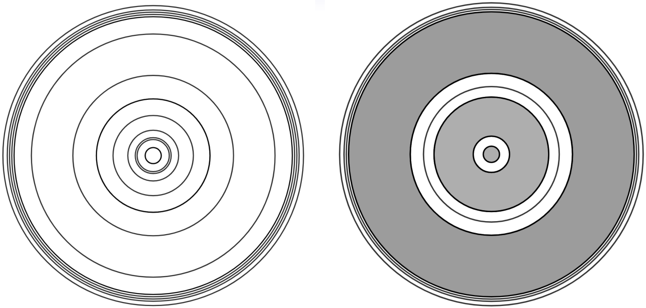

Theorem

Suppose X is an effectively compact subset of $\mathbb{R}^n$ which is a union of concentric spheres, i.e. there is a point $x_0 \in \mathbb{R}^n$ and $R \subseteq [0, +\infty\rangle$ such that

$$X = \bigcup_{r \in R} S(x_0, r).$$

Then X is computably categorical.

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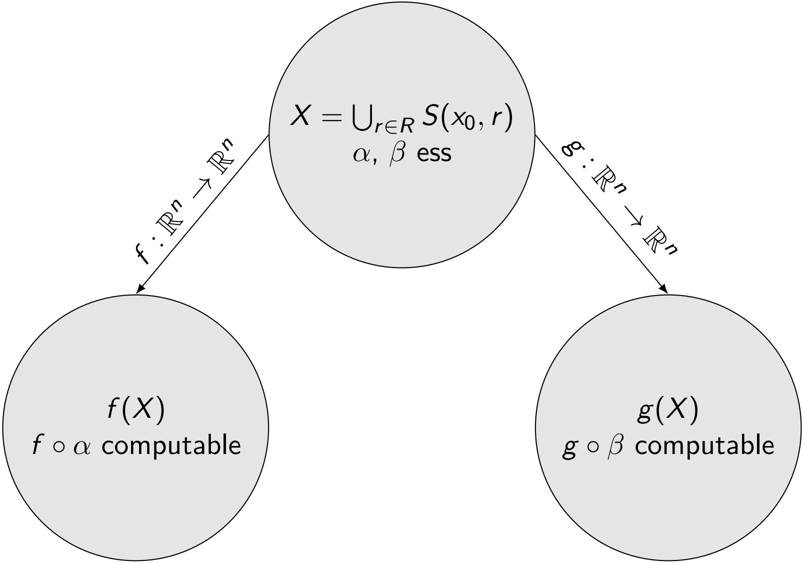
(3) In an effectively compact computable metric space any co-c.e. topological sphere is computable.

Sketch of the proof



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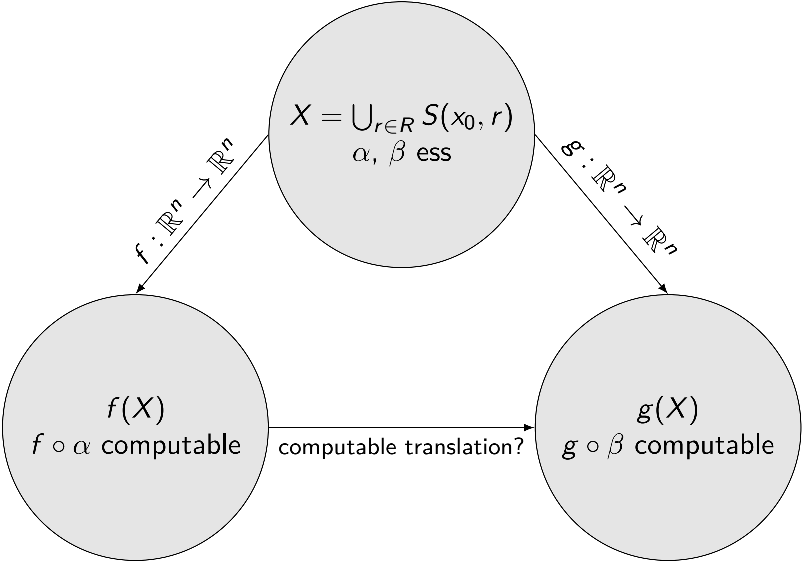

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x a computable point in $(X, d, \alpha) \Longrightarrow f(x)$ a computable point in $(f(X), d, f \circ \alpha)$

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x a computable point in $(X, d, \alpha) \implies f(x)$ a computable point in $(f(X), d, f \circ \alpha)$

$\mathrm{Orb}(f(x)) = S(f(x_0), r)$ is a co-c.e. sphere in a computable set $f(X)$

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$t(x) = x - f(x_0) + g(x_0)$ is a computable translation which maps $f(X)$ to $g(X)$

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Computable categoricity in $\mathbb{R}^2$

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Conclusion for $\mathbb{R}^2$:

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If X is a compact subset of $\mathbb{R}^2$ which has infinitely many isometries, then X is a union of concentric circles.

Conclusion for $\mathbb{R}^2$:

Theorem

Any compact subset of $\mathbb{R}^2$ which admits a structure of an effectively compact computable metric space is computably categorical.

Computable categoricity of sets with a line of symmetry

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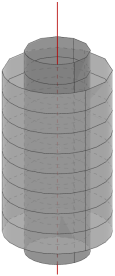

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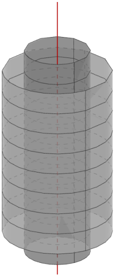

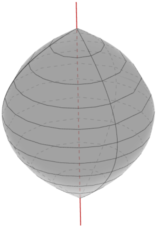

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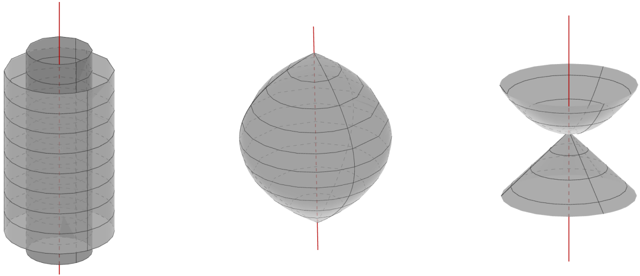

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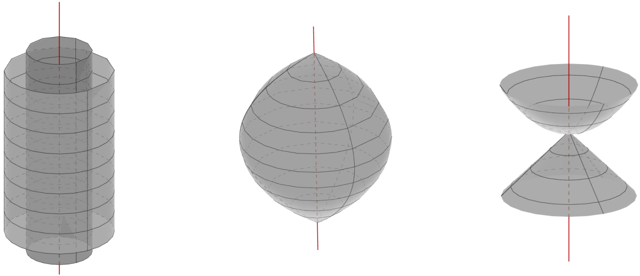
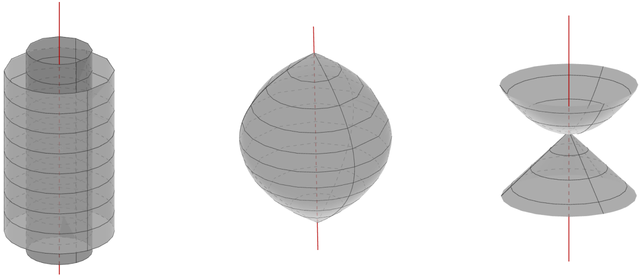


p is the **line of symmetry** of $X \subseteq \mathbb{R}^3$ if $X = \bigcup_{i \in I} S_i$,
where $\{S_i \mid i \in I\}$ are circles with centers on the line p which lie in parallel planes perpendicular to p.

Computable categoricity of sets with a line of symmetry



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Theorem

Suppose $X \subseteq \mathbb{R}^3$ is an effectively compact metric space with the line of symmetry p. Then X is computably categorical.

Key facts used in the proof

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(1) Assume that $X \subseteq \mathbb{R}^n$, d Euclidean metric on X, (X, d, α) effectively compact. Then there exists an isometry f of $\mathbb{R}^n$ such that $f \circ \alpha$ is a computable sequence in $\mathbb{R}^n$. For that isometry f, the set $f(X)$ is computable in $\mathbb{R}^n$.

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Then $\alpha \sim \beta$.

Key facts used in the proof

(1) Assume that $X \subseteq \mathbb{R}^n$, d Euclidean metric on X, (X, d, α) effectively compact. Then there exists an isometry f of $\mathbb{R}^n$ such that $f \circ \alpha$ is a computable sequence in $\mathbb{R}^n$. For that isometry f, the set $f(X)$ is computable in $\mathbb{R}^n$.

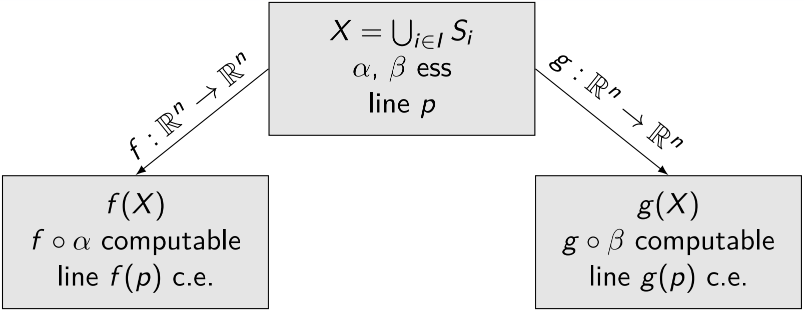
(2) Assume that α and β computable sequences in $\mathbb{R}^n$ such that

$$\overline{\{\alpha_i \mid i \in \mathbb{N}\}} = \overline{\{\beta_i \mid i \in \mathbb{N}\}}.$$

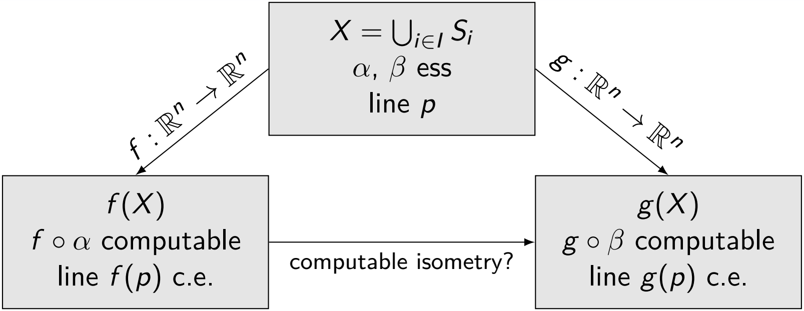
Then $\alpha \sim \beta$.

(3) If X is a computable set in $\mathbb{R}^3$ with at least two points and p is its line of symmetry, then p is computably enumerable.

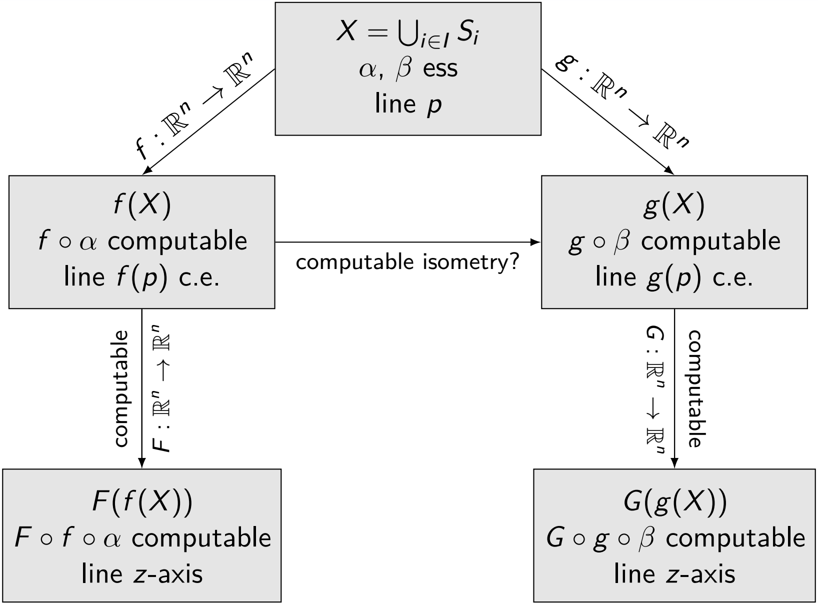
Sketch of the proof



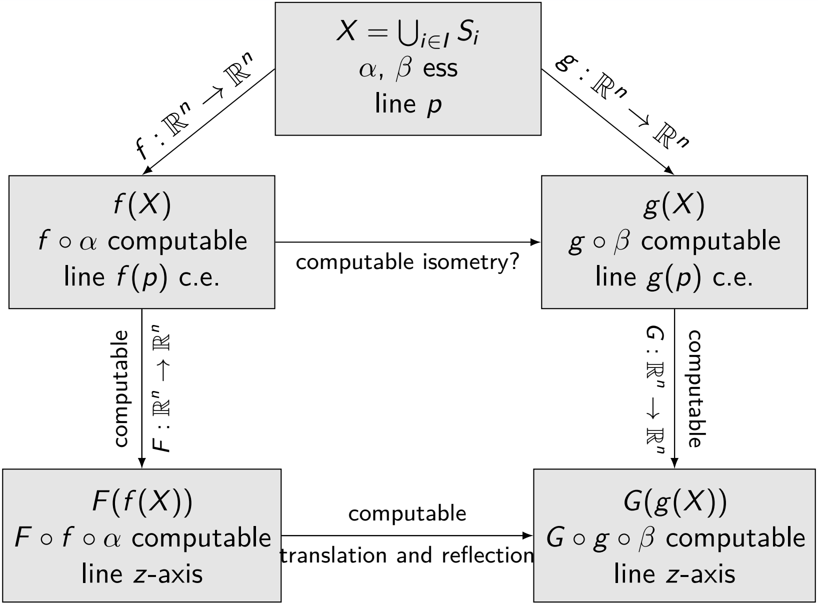
Sketch of the proof

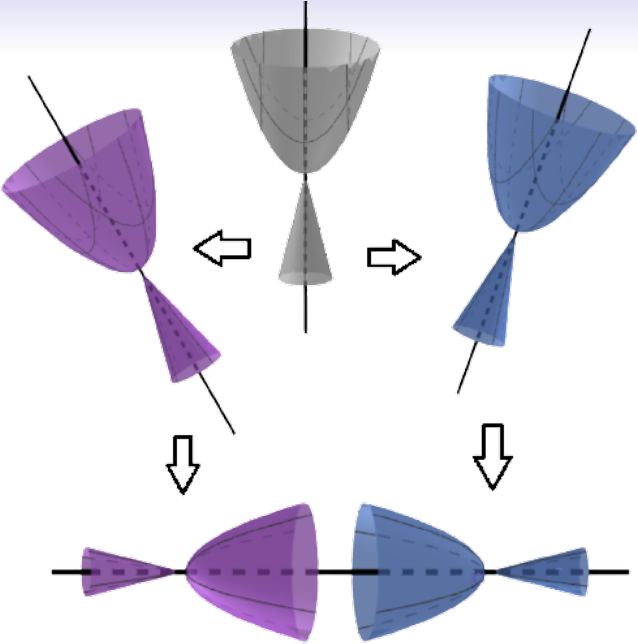


Sketch of the proof



Sketch of the proof





Future work

Future work

$$\text{effective compactness} \overset{?}{\Longrightarrow} \text{computable categoricity}$$