Globally Sound Analytic Calculi for Quantifier Macros

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analytic proof

- introduced by Gottfried Wilhelm Leibniz and Immanuel Kant
- a proof where all the information used in the proof is already contained in the end-sequent
- idealization, however cut-free complete sequent calculi can be considered since Gentzen 1934 as a close approximation

We refine the concept of analyticity by considering macros of connectives and quantifiers (used to deal with explicit definitions, handling of integrals as objects).

In logic an analytic proof of a statement containing only macros of connectives and quantifiers would itself be based on these macros.

Is it possible to form inference rules for macros that are compatible with cut-elimination?

- "yes" for macros of connectives
- "no" if macros of quantifiers are considered in the framework of usual eigenvariable conditions

In contrast an analytic framework can be constructed if globally sound but possibly locally unsound concepts of proof are introduced.

Macros for Connectives

macro for connectives

A macro for connectives is a formula based on propositional variables which is considered as a connective in its own right.

Example

The binary connective $\leftrightarrow$:

$$A \leftrightarrow B = (A \supset B) \wedge (B \supset A).$$

Proposition

To every macro for connectives $\Box$ a left inference rule, denoted by $\Box_l$, and right inference rule, denoted by $\Box_r$, are associated such that **LK** extended by $\Box_l$ and $\Box_r$ admits cut-elimination.

Proof. Simple.

Macros for Connectives

Example

For $A \leftrightarrow B = (A \supset B) \wedge (B \supset A)$ $\Box_r$ is defined via

$$\dfrac{\dfrac{A \to B}{\to A \supset B}\supset_r \qquad \dfrac{B \to A}{\to B \supset A}\supset_r}{\to (A \supset B) \wedge (B \supset A)}\wedge_r$$

and $\Box_l$ is defined via

$$\dfrac{\dfrac{\dfrac{\to A, B \qquad A \to A}{B \supset A \to A}\supset_l \qquad \dfrac{B \to B \qquad A, B \to}{B, B \supset A \to}\supset_l}{(A \supset B), (B \supset A) \to}\supset_l}{(A \supset B) \wedge (B \supset A) \to}\wedge_l$$

Macros for Connectives

Consequently, the inference rules for $\leftrightarrow$ are

$$\frac{A, \Gamma \to \Delta, B \qquad B, \Gamma \to \Delta, A}{\Gamma \to \Delta, A \leftrightarrow B} \leftrightarrow_r \quad \frac{A, B, \Gamma \to \Delta \qquad \Gamma \to \Delta, A, B}{A \leftrightarrow B, \Gamma \to \Delta} \leftrightarrow_l$$

The critical reduction step for $\leftrightarrow$ in the cut-elimination procedure of Gentzen is (contractions are hidden):

$$\frac{\dfrac{\Gamma_1 \to \Delta_1, A, B \qquad B, \Gamma_2 \to \Delta_2, A}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2, A} * \qquad \dfrac{A, \Gamma_3 \to \Delta_3, B \qquad A, B, \Gamma_4 \to \Delta_4}{A, \Gamma_3, \Gamma_4 \to \Delta_3, \Delta_4} *}{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \to \Delta_1, \Delta_2, \Delta_3, \Delta_4} \mathit{cut}$$

where $*$ is the *mix* rule.

Macros for Quantifiers

macro for quantifiers

A macro for quantifiers M is a formula based on quantifiers $Q_i \in \{\forall, \exists\}, 1 \leq i \leq n$ which is considered as a connective in its own right:

$$M_{x_1,\ldots,x_n} A(x_1, \ldots, x_n) = Q_1 x_1, \ldots, Q_n x_n A(x_1, \ldots, x_n).$$

Example

The quantifier macro Q:

$$Q_{x,y} A(x, y) = \forall x \exists y A(x, y).$$

The language $\mathcal{L}_Q$ is based on the usual language of first-order logic with exception that the quantifiers are replaced by the quantifier Q.

The Calculus **LQ**: A First Approach

The calculus **LQ** is **LK**, where the quantifier rules are exchanged by

$$\frac{\Gamma \to \Delta, A(a,t)}{\Gamma \to \Delta, Q_{x,y}A(x,y)}\; Q_r \qquad\qquad \frac{A(t',a'), \Gamma \to \Delta}{Q_{x,y}A(x,y), \Gamma \to \Delta}\; Q_l$$

where a does not occur in the lower sequent and a' does not occur in the lower sequent and in t'.

The inferences Q_r and Q_l are derived from

$$\frac{\dfrac{\Gamma \to \Delta, A(a,t)}{\Gamma \to \Delta, \exists y A(a,y)}\;\exists_r}{\Gamma \to \Delta, \forall x \exists y A(x,y)}\;\forall_r \qquad\qquad \frac{\dfrac{A(t,a), \Gamma \to \Delta}{\exists y A(t,y), \Gamma \to \Delta}\;\exists_l}{\forall x \exists y A(x,y), \Gamma \to \Delta}\;\forall_l$$

The Calculus **LQ**: A First Approach

The dual quantifier Q^D to Q can be defined in the usual dual way

$$Q^D_{x,y}A(x,y) = \neg Q_{x,y}\neg A(x,y) = \exists x \forall y \neg A(x,y).$$

The quantifier introduction rules for Q^D are

$$\frac{\Gamma \to \Delta, A(t,a)}{\Gamma \to \Delta, Q^D_{x,y}A(x,y)}\ Q^D_r$$

where a does not occur in the lower sequent and in t and

$$\frac{A(a,t), \Gamma \to \Delta}{Q^D_{x,y}A(x,y), \Gamma \to \Delta}\ Q^D_l$$

where a does not occur in the lower sequent.

The usual quantifier rules of **LK** can be obtained by partial dummy applications of Q.

The Calculus **LQ**: A First Approach

Example

The sequent $Q_{x,y}A(x,y) \to \forall x \exists y A(x,y)$ is derivable in **LQ**:

$$\dfrac{\dfrac{\dfrac{A(a,b) \to A(a,b)}{A(a,b) \to \exists y A(a,y)}\ \exists_r}{Q_{x,y}A(x,y) \to \exists y A(a,y)}\ Q_l}{Q_{x,y}A(x,y) \to \forall x \exists y A(x,y)}\ \forall_r$$

Theorem

LQ is sound.

Proof. The macro Q can be replaced by $\forall\exists$ everywhere in the derivation. The resulting derivation is an **LK**-derivation.

The Calculus **LQ**: A First Approach

Theorem

LQ admits cut-elimination.

Proof. The only difference to Gentzen's proof is the reduction of Q, which can be performed as follows:

$$\dfrac{\dfrac{\Gamma \to \Delta, A(a,t)}{\Gamma \to \Delta, Q_{x,y}A(x,y)}\,Q_r \qquad \dfrac{A(t',a'), \Pi \to \Lambda}{Q_{x,y}A(x,y), \Pi \to \Lambda}\,Q_l}{\Gamma, \Pi \to \Delta, \Lambda}\,cut$$

(all occurrences of a, a', t, t' are indicated). This can be reduced to

$$\dfrac{\Gamma \to \Delta, A(t',t) \qquad A(t',t), \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}\,cut$$

Corollary

Mid-sequent theorem.

The Calculus **LQ**: A First Approach

Proposition

LQ is incomplete w.r.t. the sequents provable in **LK**.

Proof. Assume by contradiction the sequent

$$Q_{x,y}A(x,y) \rightarrow Q_{x,y}(A(x,y) \vee C)$$

is provable. Then it is provable without cuts. A cut-free derivation after deletion of weakenings and contractions has the initial form

$$\dfrac{\dfrac{A(a,b) \rightarrow A(a,b)}{A(a,b) \rightarrow A(a,b) \vee C}}{\vdots}$$

Due to the mixture of strong (eigenvariable dependent) and weak positions in Q none of the inference rules Q_r, Q_l can be applied.

Corollary

Compound axioms $A \rightarrow A$ cannot be reduced to atomic ones.

The Calculus **LQ**: A First Approach

The usual quantifier shifts of classical logic are not derivable in **LQ**. Let $Q^* \in \{Q, Q^D\}$ and $\circ \in \{\wedge, \vee\}$. Then the quantifier shifts for the operators $\wedge, \vee$ are:

1. $Q^*_{x,y}(A \circ B(x,y)) \to A \circ Q^*_{x,y}B(x,y)$,
2. $Q^*_{x,y}(A(x,y) \circ B) \to Q^*_{x,y}A(x,y) \circ B$.

Let $(Q^*, Q^{D^*}) \subseteq \{(Q, Q^D), (Q^D, Q)\}$, then the quantifier shifts for $\supset$ are:

3. $Q^*_{x,y}(A \supset B(x,y)) \to A \supset Q^*_{x,y}B(x,y)$,
4. $Q^*_{x,y}(A(x,y) \supset B) \to Q^{D^*}_{x,y}A(x,y) \supset B$,
5. $A \supset Q^*_{x,y}B(x,y) \to Q^*_{x,y}(A \supset B(x,y))$,
6. $Q^{D^*}_{x,y}A(x,y) \supset B \to Q^*_{x,y}(A(x,y) \supset B)$.

The quantifier shifts for $\neg$ are:

7. $Q^*_{x,y}\neg A(x,y) \to \neg Q^{D^*}_{x,y}A(x,y)$,
8. $\neg Q^*_{x,y}A(x,y) \to Q^{D^*}_{x,y}\neg A(x,y)$.

The inherent incompleteness of **LQ** for trivial statements is a consequence of the fact that Q represents a quantifier inference macro combining a strong and a weak quantifier.

Solution: consider sequent calculi with concepts of proof which are globally but not locally sound.
This means that all derived statements are true but that not every subderivation is meaningful.

(1) That Kurt Gödel is Austrian entails that Kurt Gödel is Austrian.
(2) Hence, that Kurt Gödel is Austrian entails that everyone is Austrian.
(3) That is, if Kurt Gödel is Austrian, then all people are Austrian.
(4) Therefore, there exists a person such that, if that person is Austrian, then all people are Austrian.

$$
\dfrac{\dfrac{\dfrac{A(a) \rightarrow A(a)}{A(a) \rightarrow \forall y A(y)}}{\rightarrow A(a) \supset \forall y A(y)}}{\rightarrow \exists x (A(x) \supset \forall y A(y))}
$$

The traditional way to ensure soundness

- Inferences are sound, i.e. only true conclusions result from true premises.
- Derivations are hereditary, i.e. initial segments of proofs are proofs themselves.

Weak regularity

$$
\dfrac{\dfrac{\dfrac{A(a) \to A(a)}{A(a) \to \forall x A(x)} \qquad \dfrac{A(f(a)) \to A(f(a))}{\forall x A(x) \to A(f(a))}}{\dfrac{A(a) \to A(f(a))}{\to A(a) \supset A(f(a))}}}{\to \exists x (A(x) \supset A(f(x)))}
$$

$\mathbf{LK}^{+}$ and $\mathbf{LK}^{++}$

side variable relation $<_{\varphi,\mathbf{LK}}$

Let φ be an **LK**-derivation. We say b is a side variable of a in φ (written $a <_{\varphi,\mathbf{LK}} b$) if φ contains a strong quantifier inference of the form

$$\frac{\Gamma \to \Delta, A(a, b, \overline{c})}{\Gamma \to \Delta, \forall x A(x, b, \overline{c})}\ \forall_r$$

or of the form

$$\frac{A(a, b, \overline{c}), \Gamma \to \Delta}{\exists x A(x, b, \overline{c}), \Gamma \to \Delta}\ \exists_l$$

$\mathbf{LK}^+$ and $\mathbf{LK}^{++}$

LK-suitable quantifier inferences

We say a quantifier inference is suitable for a proof φ if either it is a weak quantifier inference, or the following three conditions are satisfied:

- (substitutability) the eigenvariable does not appear in the conclusion of φ.
- (side variable condition) the relation $<_{\varphi,\mathbf{LK}}$ is acyclic.
- (weak regularity) the eigenvariable of an inference is not the eigenvariable of another inference in φ.

$\mathbf{LK}^+$

We obtain $\mathbf{LK}^+$ from **LK** by replacing the usual eigenvariable conditions by **LK**-suitable ones.

$\mathbf{LK}^{+}$ and $\mathbf{LK}^{++}$

LK-weakly suitable quantifier inference

A quantifier inference is *weakly suitable for a proof* φ if either it is a weak quantifier inference or it satisfies substitutability, the side-variable condition, and:

- (very weak regularity) the eigenvariable of an inference with main formula A is different to the eigenvariable of an inference with main formula A' whenever $A \neq A'$.

$\mathbf{LK}^{++}$

We obtain $\mathbf{LK}^{++}$ from **LK** by replacing the usual eigenvariable conditions by **LK**-weakly suitable ones.

Soundness

Theorem.
If a sequent is $\mathbf{LK}^{++}$-derivable, then it is already $\mathbf{LK}$-derivable.

Proof. Let π be an $\mathbf{LK}^{++}$-proof. Replace every unsound universal quantifier inference by an $\supset L$ inference:

$$\frac{\Gamma \to \Delta, A(a) \qquad \forall x A(x) \to \forall x A(x)}{\Gamma, A(a) \supset \forall x A(x) \to \Delta, \forall x A(x)}$$

Similarly replace every unsound existential quantifier by an $\supset L$ inference

$$\frac{\exists x A(x) \to \exists x A(x) \qquad A(a), \Gamma \to \Delta}{\Gamma, \exists x A(x), \exists x A(x) \supset A(a) \to \Delta}$$

By doing this, we obtain a proof of the desired sequent, together with many formulae of the form $A(a) \supset \forall x A(x)$ or $\exists x A(x) \supset A(a)$ on the left-hand side. Introduce existential quantifiers left. This is sound in $\mathbf{LK}$ by properties of $<_\pi$.

Corollary.

If a sequent is derivable in $\mathbf{LK}^{+}$ or $\mathbf{LK}^{++}$, then it is already derivable in $\mathbf{LK}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A(a,b) \vdash A(a,b)}{A(a,b) \vdash \forall y A(a,y)}}{A(a,b) \vdash \exists x \forall y A(x,y)}}{\exists x A(x,b) \vdash \exists x \forall y A(x,y)}}{\forall y \exists x A(x,y) \vdash \exists x \forall y A(x,y)}$$

$$a <_{\pi} b \quad b <_{\pi} a \,!$$

LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \to A(a)}{A(a) \to A(a), B}}{\to A(a), A(a) \supset B}}{\to A(a), \exists x\,(A(x) \supset B)}}{\to \exists x\,(A(x) \supset B), A(a)}}{\to \exists x\,(A(x) \supset B), \forall x\, A(x)} \qquad B \to B}{\forall x\, A(x) \supset B \to \exists x\,(A(x) \supset B), B}}{\forall x\, A(x) \supset B, A(b) \to \exists x\,(A(x) \supset B), B}}{\forall x\, A(x) \supset B \to \exists x\,(A(x) \supset B), A(b) \supset B}}{\dfrac{\forall x\, A(x) \supset B \to \exists x\,(A(x) \supset B), \exists x\,(A(x) \supset B)}{\forall x\, A(x) \supset B \to \exists x\,(A(x) \supset B)}}
$$

LK^{+}

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a) \to A(a)}{A(a) \to \forall x\, A(x)} \qquad B \to B}{A(a), \forall x\, A(x) \supset B \to B}}{\forall x\, A(x) \supset B, A(a) \to B}}{\forall x\, A(x) \supset B \to A(a) \supset B}}{\forall x\, A(x) \supset B \to \exists x\, (A(x) \supset B)}
$$

Theorem.
There is no elementary function bounding the length of the shortest cut-free **LK**-proof of a formula in terms of its shortest cut-free $\mathbf{LK}^{+}$-proof.

An immediate consequence is the following:

Corollary.
There is no elementary function bounding the length of the shortest cut-free **LK**-proof of a formula in terms of its shortest cut-free $\mathbf{LK}^{++}$-proof.

$\mathbf{LK}^{+}$ and $\mathbf{LK}^{++}$

Example

Consider the following locally unsound but globally sound derivation φ in $\mathbf{LK}^{+}$ (and $\mathbf{LK}^{++}$):

$$\dfrac{\dfrac{\dfrac{A(a) \rightarrow A(a)}{A(a) \rightarrow \forall y A(y)}\ \forall_r}{\rightarrow A(a) \supset \forall y A(y)}\ \supset_r}{\rightarrow \exists x (A(x) \supset \forall y A(y))}\ \exists_r$$

The side variable relation $<_{\varphi,\mathbf{LK}}$ is empty.

The Analytic Sequent Calculus $\mathbf{LQ}^{++}$

side variable relation $<_{\varphi,\mathbf{LQ}}$

Let φ be an **LQ**-derivation. We say b is a side variable of a in φ (written $a <_{\varphi,\mathbf{LQ}} b$) if φ contains a strong quantifier inference of the form

$$\frac{A(t,a), \Gamma \to \Delta}{Q_{x,y}A(x,y), \Gamma \to \Delta}\ Q_l$$

and b occurs in t.

The Analytic Sequent Calculus $\mathbf{LQ}^{++}$

LQ-weakly suitable quantifier inferences

A quantifier inference is **LQ**-suitable for a proof φ if the following three conditions are satisfied:

- (substitutability) the eigenvariable does not appear in the conclusion of φ.
- (side variable condition) the relation $<_{\varphi,\mathbf{LQ}}$ is acyclic.
- (very weak regularity) the eigenvariable of an inference with main formula A is different to the eigenvariable of an inference with main formula A' whenever $A \neq A'$.

analytic sequent calculus $\mathbf{LQ}^{++}$

The sequent calculus $\mathbf{LQ}^{++}$ is **LQ**, except that we replace quantifier inferences with **LQ**-weakly suitable quantifier inferences.

The Analytic Sequent Calculus $\mathbf{LQ}^{++}$

Example

The sequent $Q_{x,y}A(x,y) \rightarrow Q_{x,y}(A(x,y) \vee C)$ is $\mathbf{LQ}^{++}$-derivable. Consider the derivation $\varphi =$

$$\dfrac{\dfrac{\dfrac{A(a,b) \rightarrow A(a,b)}{A(a,b) \rightarrow A(a,b) \vee C}\; w_r + \vee_r}{A(a,b) \rightarrow Q_{x,y}(A(x,y) \vee C)}\; Q_r}{Q_{x,y}A(x,y) \rightarrow Q_{x,y}(A(x,y) \vee C)}\; Q_l$$

with $b <_{\varphi,\mathbf{LQ}} a$.

The Analytic Sequent Calculus $\mathbf{LQ}^{++}$

In contrast to $\mathbf{LQ}$ the usual quantifier shifts are derivable in $\mathbf{LQ}^{++}$.

The quantifier shift $Q_{x,y}A(x,y) \supset B \rightarrow Q^D_{x,y}(A(x,y) \supset B)$ is derivable in $\mathbf{LQ}^{++}$. Its derivation is $\varphi =$

$$\dfrac{\dfrac{\dfrac{\dfrac{A(a,b) \rightarrow A(a,b)}{A(a,b) \rightarrow Q_{x,y}A(x,y)}\,Q_r \qquad B \rightarrow B}{A(a,b), Q_{x,y}A(x,y) \supset B \rightarrow B}\,\supset_l}{Q_{x,y}A(x,y) \supset B \rightarrow A(a,b) \supset B}\,\supset_r}{Q_{x,y}A(x,y) \supset B \rightarrow Q^D_{x,y}(A(x,y) \supset B)}\,Q^D_r$$

with $b <_{\varphi,\mathbf{LQ}} a$.

Cut-Elimination for $\mathbf{LQ}^{++}$

To show cut-elimination for $\mathbf{LQ}^{++}$ we will translate $\mathbf{LQ}^{++}$-derivations into cut-free $\mathbf{LK}$-derivations and vice versa.

Let S_Q be an $\mathbf{LQ}$-sequent. Then $S_{\forall\exists} = S_Q\{Q_{x,y} \leftarrow \forall x \exists y\}$.
Let $S_{\forall\exists}$ be an $\mathbf{LK}$-sequent containing quantifier occurrences only in blocks of the form $\forall x \exists y$. Then $S_Q = S_{\forall\exists}\{\forall x \exists y \leftarrow Q_{x,y}\}$.

Cut-Elimination for $\mathbf{LQ}^{++}$

Lemma

An $\mathbf{LQ}^{++}$-derivation φ of S_Q can be transformed into a cut-free $\mathbf{LK}$-derivation from atomic axioms of $S_{\forall\exists}$.

Proof. By translating Q to $\forall\exists$ we obtain an $\mathbf{LK}^{++}$-derivation which can be transformed into an $\mathbf{LK}$-derivation. As $\mathbf{LK}$ admits cut-elimination we obtain a cut-free $\mathbf{LK}$-derivation. Compound axioms in $\mathbf{LK}$ can be replaced by atomic ones.

Cut-Elimination for $\mathbf{LQ}^{++}$

Lemma

A cut-free **LK**-derivation of a sequent $S_{\forall\exists}$ containing quantifiers only in the form of blocks $\forall x \exists y$ can be transformed into a cut-free $\mathbf{LQ}^{++}$-derivation of S_Q.

Proof By permuting inferences:

- $\exists_l$ occurs: infer $\forall_l$ immediately afterwards (this has no impact on the result nor on the proof being cut-free).
- $\forall_r$ with principal formula $\forall x \exists y A(x, y)$ and eigenvariable a: determine all existential inferences with principal formula $\exists y A(a, y)$ and introduce $\forall x \exists y A(x, y)$ immediately after these inferences (the original **LK**-derivation is regular).
- We obtain an $\mathbf{LQ}^{++}$-derivation: very weak regularity holds, the eigenvariables do not occur in the end-sequent and $<_{\varphi,\mathbf{LQ}}$ does not loop as the order on the inferences is respected.

Why not $\mathbf{LQ}^+$?

$\mathbf{LQ}^+$ is not a complete calculus, because the sequent

$$\forall x(A(x, c_1) \vee A(x, c_2)) \to Q_{x,y}A(x, y)$$

is not $\mathbf{LQ}^+$-derivable. However, it is $\mathbf{LQ}^{++}$-derivable:

$$\dfrac{\dfrac{\dfrac{A(a, c_1) \to A(a, c_1)}{A(a, c_1) \to Q_{x,y}A(x, y)}\,Q_r \qquad \dfrac{A(a, c_2) \to A(a, c_2)}{A(a, c_2) \to Q_{x,y}A(x, y)}\,Q_r}{A(a, c_1) \vee A(a, c_2) \to Q_{x,y}A(x, y)}\,\vee_l}{\forall x(A(x, c_1) \vee A(x, c_2)) \to Q_{x,y}A(x, y)}\,\forall_l$$

Quantifier shifts not valid intuitionistically

1. $\forall x\,(A \vee B(x)) \to A \vee \forall x\, B(x)$;
2. $(\forall x\, A(x) \supset B) \to \exists x\,(A(x) \supset B)$;
3. $(A \supset \exists x\, B(x)) \to \exists x\,(A \supset B(x))$.

Proposition.

A sequent is provable in LJ^{++} if and only if it is provable in LJ with all quantifier shifts added as axioms.

No elementary Skolemization for cut-free $\mathbf{LK}^{+}$ and $\mathbf{LK}^{++}$ proofs. (But quadratic Skolemization using additional cuts.)

No elementary extraction of Skolemized Herbrand disjunctions from cut-free $\mathbf{LK}^{+}$ and $\mathbf{LK}^{++}$ proofs.

No Gentzen-style cut-elimination (as Gentzen-style cut-elimination would transform LJ^{+} (LJ^{++}) proofs into cut-free LJ^{+} (LJ^{++}) proofs).

Relation to the ε-calculus

$$\exists x A(x) \sim A(\varepsilon_x A(x))$$

$$\forall x A(x) \sim A(\varepsilon_x \neg A(x)) \sim A(\tau_x A(x))$$

LK_ε

$$\frac{\Gamma, A(t) \to \Delta}{\Gamma, A(\tau_x A(x)) \to \Delta}\,\tau$$

$$\frac{\Gamma \to \Delta, A(t)}{\Gamma \to \Delta, A(\varepsilon_x A(x))}\,\varepsilon$$

Relation to the ε-calculus

Another soundness proof for $\mathbf{LK}^{+}$ and $\mathbf{LK}^{++}$
But e.g.

$$\begin{array}{c} (\varphi) \\ \Gamma \to \Delta, A(s(t)) \\ \hline \Gamma \to \Delta, A(s(\varepsilon_x A(s(x)))) \\ \hline\hline \Gamma' \to \Delta', A(s(\varepsilon_x A(s(x)))) \\ \hline \Gamma' \to \Delta', A(\varepsilon_x A(x)) \end{array}$$

Not represented in $\mathbf{LK}^{+}$ and $\mathbf{LK}^{++}$.

Henkin Quantifiers

A formula A using Q_H can be written as

$$A_H = \begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x, y, u, v).$$

$\exists f \exists g \forall x \forall y A(x, y, f(x), g(y))$ where f and g are function variables.

Analytic Sequent Calculus **LH**: A First Approach

Definition

- Axioms: $A \to A$
- The rules of **LH** are as the rules of **LK**, except that we replace the quantifier rules by:

$$\frac{\Gamma \to \Delta, A(a, b, t_1, t_2)}{\Gamma \to \Delta, \begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x, y, u, v)} \; Q_{H\,r}$$

a and b are eigenvariables ($a \neq b$) not allowed to occur in the lower sequent and t_1 and t_2 are terms such that t_1 must not contain b and t_2 must not contain a.

Analytic Sequent Calculus **LH**: A First Approach

$$\frac{A(t'_1, t'_2, a, b), \Pi \to \Gamma}{\begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x, y, u, v), \Pi \to \Gamma} \; Q_{H I_1}$$

where a and b are eigenvariables ($a \neq b$) not allowed to occur in the lower sequent and t'_1, t'_2 are terms such that b does not occur in t'_2 and a and b do not occur in t'_1.

$$\frac{A(t'_1, t'_2, a, b), \Pi \to \Gamma}{\begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x, y, u, v), \Pi \to \Gamma} \; Q_{H I_2}$$

where a and b are eigenvariables ($a \neq b$) not allowed to occur in the lower sequent and t'_1, t'_2 are terms such that a does not occur in t'_1 and a and b do not occur in t'_2.
$\forall$ and $\exists$ partial dummy applications of Q_H.

The rule Q_{H_r} originates from an analysis of a corresponding inference sequence in a suitable partial second-order calculus for functions:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \to \Delta, A(a, b, s(a), t(b))}{\Gamma \to \Delta, \forall y A(a, y, s(a), t(y))}}{\Gamma \to \Delta, \forall x \forall y A(x, y, s(x), t(y))}}{\Gamma \to \Delta, \exists g \forall x \forall y A(x, y, s(x), g(y))}}{\Gamma \to \Delta, \exists f \exists g \forall x \forall y A(x, y, f(x), g(y))}$$

The rules $Q_{H_{l_1}}$ and $Q_{H_{l_2}}$ originate from

$$\dfrac{\dfrac{\dfrac{\dfrac{A(t, t', f'(t), g'(t')), \Gamma \to \Delta}{\forall y A(t, y, f'(t), g'(y)), \Gamma \to \Delta}}{\forall x \forall y A(x, y, f'(x), g'(y)), \Gamma \to \Delta}}{\exists g \forall x \forall y A(x, y, f'(x), g(y)), \Gamma \to \Delta}}{\exists f \exists g \forall x \forall y A(x, y, f(x), g(y)), \Gamma \to \Delta}$$

with eigenvariables f' and g'. $f'(t)$ can obviously not occur in t and $g'(t')$ can obviously not occur in t'. $f'(t)$ either does not occur in t' of $g'(t')$ does not occur in t.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a,b,c,d) \to A(a,b,c,d)}{A(a,b,c,d) \to \exists v A(a,b,c,v)}\,\exists_r}{A(a,b,c,d) \to \exists u \exists v A(a,b,u,v)}\,\exists_r}{\begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x,y,u,v) \to \exists u \exists v A(a,b,u,v)}\,Q_{H_l}}{\begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x,y,u,v) \to \forall y \exists u \exists v A(a,y,u,v)}\,\forall_r}{\begin{pmatrix} \forall x & \exists u \\ \forall y & \exists v \end{pmatrix} A(x,y,u,v) \to \forall x \forall y \exists u \exists v A(x,y,u,v)}\,\forall_r$$