Computable sequences filling computable sets in computable topological spaces

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Motivation

In topology, important concept is one of separability: having a countable dense subset.

Definition

A topological space $(X, \mathcal{T})$ is *separable* if there is a sequence $(y_i)_{i \in \mathbb{N}}$ such that every $\emptyset \neq U \in \mathcal{T}$ contains some y_i from it.

Clearly, if $(X, \mathcal{T})$ is separable and $S \subseteq X$ is closed and nonempty, then there is a sequence $(x_i)_{i \in \mathbb{N}}$ such that $\overline{\{x_i : i \in \mathbb{N}\}} = S$: just re-enumerate those y_i that fall into S.

What about computability: if the ambient space is computable, and set S is computable, can we find a computable $(x_i)_i$? Yes!

Computable topological space

Definition

Let $(X, \mathcal{T})$ be a topological space and let I be a sequence in $\mathcal{T}$ whose image is a basis for $\mathcal{T}$.
$(X, \mathcal{T}, I)$ is a *computable topological space* if there exist computably enumerable (*c.e.*) relations $\mathcal{C}, \mathcal{D} \subseteq \mathbb{N}^2$ such that:

1. $i \, \mathcal{C} \, j \Longrightarrow I_i \subseteq I_j$;
2. $i \mathcal{D} j \Rightarrow I_i \cap I_j = \emptyset$;
3. $x \in I_i \cap I_j \Longrightarrow \exists k(x \in I_k \wedge k \, \mathcal{C} \, i \wedge k \, \mathcal{C} \, j)$;
4. $x \neq y \Longrightarrow \big(\exists (k, l) \in \mathcal{D}\big)(x \in I_k \wedge y \in I_l)$.

We can also stipulate useful properties such as reflexivity and transitivity of $\mathcal{C}$, and $i \, \mathcal{C} \, j \, \mathcal{D} \, k \Longrightarrow i \, \mathcal{D} \, k$.

We enumerate finite nonempty sets $\big\{(j)_0, (j)_1, \dots, (j)_{\bar{j}}\big\} =: [j]$ and finite nonempty sequences $\big((j)_0, (j)_1, \dots, (j)_{\bar{j}}\big)$ of natural numbers.

Finite unions of base open sets

$$J_j := \bigcup_{i \in [j]} I_i$$

$$\mathcal{J} := \{J_j : j \in \mathbb{N}\}$$

$$J_j \subseteq_\forall I_{i'} :\Longleftrightarrow (\forall i \in [j])(i \,\mathcal{C}\, i')$$
$$I_i \subseteq_\exists J_{j'} :\Longleftrightarrow (\exists i' \in [j'])(i \,\mathcal{C}\, i')$$
$$J_j \subseteq_\forall J_{j'} :\Longleftrightarrow (\forall i \in [j])(I_i \subseteq_\exists J_{j'})$$
$$J_j \subseteq_\exists J_{j'} :\Longleftrightarrow (\exists i' \in [j'])(J_j \subseteq_\forall I_{i'})$$
$$J_j \diamond J_j' :\Leftrightarrow (\forall i \in [j], i' \in [j'])(i \,\mathcal{D}\, i')$$

Lemma (expansion lemma)

Let $\mathcal{F}$ be a finite family of compact sets and $A \subseteq \mathcal{J}$ finite. Then there exists a family $(J_K)_{K \in \mathcal{F}} : \mathcal{F} \to \mathcal{J}$ such that for all $K \in \mathcal{F}$:

1. $K \subseteq J_K$;
2. *if $L \in \mathcal{F}$ is disjoint with K, then $J_K \diamond J_L$;*
3. *if $K \subseteq J$ for some $J \in A$, then $J_K \subseteq_\forall J$.*

Computable sets, points and sequences

Definition

Let $(X, \mathcal{T}, I)$ be a computable topological space.

- $x \in X$ is a *computable point* in $(X, \mathcal{T}, I)$ if '$x \in I_i$' is a c.e. property of i;
- $(x_i)_{i \in \mathbb{N}} : \mathbb{N} \to X$ is a *computable sequence* in $(X, \mathcal{T}, I)$ if '$x_i \in I_j$' is a c.e. relation between i and j;
- $S \subseteq X$ is a *computably enumerable set* (*c.e. set*) in $(X, \mathcal{T}, I)$ if S is closed in $(X, \mathcal{T})$ and '$I_i \cap S \neq \emptyset$' is a c.e. property of i;
- $S \subseteq X$ is a *semicomputable set* in $(X, \mathcal{T}, I)$ if S is compact in $(X, \mathcal{T})$ and '$S \subseteq J_j$' is a c.e. property of j;
- $S \subseteq X$ is a *computable set* in $(X, \mathcal{T}, I)$ if S is both c.e. and semicomputable in $(X, \mathcal{T}, I)$.

Computable metric spaces

Definition

Let (X, d) be a metric space and let α be a sequence in X whose image is dense in (X, d). (X, d, α) is a *computable metric space* if $d(\alpha_i, \alpha_j)$ is a computable function of i and j.

Example

$(\mathbb{R}^n, d_{\text{Euclid}}, \alpha)$, where α is an effective enumeration of $\mathbb{Q}^n$.

Theorem

If (X, d, α) is a computable metric space, then $\left(X, \mathcal{T}_d, (I_i)_i\right)$ is a computable topological space, where $\mathcal{T}_d$ is induced topology, and $(I_i)_i$ enumerates all open balls around αs with rational radius.

All the definitions from the previous slide therefore make sense in computable metric spaces, too.

Computable sequence characterization

Theorem

Let (X, d, α) be a computable metric space. A sequence $(x_i)_i$ in X is computable in (X, d, α) if there exists a recursive function $f : \mathbb{N}^2 \to \mathbb{N}$ such that

$$d\big(x_i, \alpha_{f(i,k)}\big) < 2^{-k} \quad \text{for all } i, k \in \mathbb{N}.$$

Theorem

If $\mathcal{X}$ is a computable metric space and S a nonempty c.e. complete set in $\mathcal{X}$, then there exists a computable sequence in $\mathcal{X}$ whose image is dense in S.

These are already known. Notice that we only require computable enumerability and metric completeness of S.

Hausdorff closeness

Definition

Let (X, d, α) be a computable metric space, $A, B \subseteq X$ and $\varepsilon > 0$.

$$A \preceq_\varepsilon B :\Longleftrightarrow (\forall a \in A)(\exists b \in B)\big(d(a, b) < \varepsilon\big)$$

$\approx_\varepsilon$ is the symmetric closure of $\preceq_\varepsilon$. Let $\Lambda_j := \{\alpha_i : i \in [j]\}$.
Obviously $(\Lambda_j)_{j\in\mathbb{N}}$ enumerates all nonempty finite sets of αs.

Theorem

Let (X, d, α) be a computable metric space, and $\emptyset \neq S \subseteq X$. Then S is computable in (X, d, α) if and only if there exists a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$, we have

$$S \approx_{2^{-k}} \Lambda_{f(k)} .$$

Main theorem

Theorem

Let $\mathcal{X}$ be a computable topological space and let S be a nonempty computable set in $\mathcal{X}$. Then there exists a computable sequence in $\mathcal{X}$ whose image is a dense subset of S.

We sketch the proof on the next few slides.

Metrizability of S

$$S \text{ computable } \implies S \text{ semicomputable } \implies S \text{ compact}$$

$$X \text{ computable topological space} \implies X \text{ Hausdorff}$$

$$\implies X \text{ second countable}$$

$$S \subseteq X \wedge \text{above} \implies S \text{ Hausdorff} \wedge S \text{ second countable}$$

$$S \text{ compact} \wedge S \text{ Hausdorff} \implies S \text{ normal}$$

$$S \text{ normal} \wedge S \text{ second countable} \implies S \text{ metrizable}$$

So, we can just use the known result for metric spaces on S as an ambient space?

Not so fast... We now have a metric d on S such that d induces relative topology on S, but we don't have (and probably can't always have) the α sequence.

Three recursive enumerations of various Js

Because of compactness, the whole S is covered by finitely many Is (base open sets)—therefore, a J.
Because S is semicomputable, $S \subseteq J_j$ is recursively enumerable.
So, there is a recursive enumeration h of all such js.

$$j \in \operatorname{rng} h \iff J_j \supseteq S$$

Since S is nonempty, the same J that contains it also intersects it.
Since S is c.e., $S \cap I_i \neq \emptyset$ (and so also $S \cap J_j \neq \emptyset$) is recursively enumerable. So, there is a recursive enumeration f of all such js.

$$j \in \operatorname{rng} f \iff S \cap J_j \neq \emptyset \iff (\exists i \in [j])(S \cap I_i \neq \emptyset)$$

There's also a recursive φ with $[\varphi(i)] = \{i\}$, so $J_{\varphi(i)} = I_i$ for all i.

Two recursively enumerable relations on $\mathbb{N}$

$$\Omega(p,q,l) :\Longleftrightarrow \underbrace{S \subseteq J_p \cup J_q \cup J_l}_{(\Omega 1)} \wedge \underbrace{J_p \diamond J_l}_{(\Omega 2)}$$

$$\Gamma(i,k,j) :\Longleftrightarrow \underbrace{J_{f(j)} \subseteq_\forall J_{f(i)}}_{(\Gamma 1)} \wedge \underbrace{J_{f(j)} \subseteq_\exists J_{h(k)}}_{(\Gamma 2)} \wedge$$

$$\wedge \exists p \, \exists q \Big(\underbrace{\Omega\big(p,q,f(j)\big)}_{(\Gamma 3)} \wedge \underbrace{J_q \subseteq_\forall J_{f(i)}}_{(\Gamma 4)} \Big)$$

$(\Omega 2)$ is obviously r.e. (bounded quantification and $\mathcal{D}$).
$(\Gamma 1)$, $(\Gamma 2)$ and $(\Gamma 4)$ too (bounded quantification and $\mathcal{C}$).
$(\Omega 1)$ is r.e., since it is equivalent to $union\big(p, union(q,l)\big) \in \operatorname{rng} h$,
where $[union(s,t)] = [s] \cup [t]$.
Conjunctions and projections preserve being r.e. so Ω and Γ are r.e.

Closure of next is within previous

Claim 1: $\Gamma(i,k,j) \Longrightarrow \overline{J_{f(j)}} \cap S \subseteq J_{f(i)}$

Proof.

From (Γ3/Ω1), there are p and q such that $S \subseteq J_p \cup J_q \cup J_{f(j)}$.
From (Ω2), J_p is disjoint with $J_{f(j)}$, and since those are open sets, it is also disjoint with $\overline{J_{f(j)}}$. So in fact $\overline{J_{f(j)}} \cap S \subseteq J_q \cup J_{f(j)}$.
But from (Γ1), $J_{f(j)} \subseteq J_{f(i)}$, and from ($\Gamma$4), $J_q \subseteq J_{f(i)}$. $\square$

There is always next

Claim 2: $\forall i\, \forall k\, \exists j\, \Gamma(i,k,j)$

Proof.

Take any i and k. By definition of f, $J_{f(i)}$ intersects S, say in x. By definition of h, $x \in S \subseteq J_{h(k)}$, so $(\exists u \in [h(k)])(x \in I_u = J_{\varphi(u)})$. Now $J_{\varphi(u)}$ and $J_{f(i)}$ are neighborhoods of x, so there are radii $r_{1,2}$ of balls in metric space S around x within them. Set $r := \frac{\min r_{1,2}}{3}$. $\overline{B}(x,r)$, $\overline{B}(x,2r)$ and $S \setminus B(x,2r)$ are compact; $\overline{B}(x,r)$ and $S \setminus B(x,2r)$ are disjoint. Expansion lemma gives p,q,l such that

$$\overline{B}(x,r) \subseteq J_l, \quad \overline{B}(x,2r) \subseteq J_q, \quad S \setminus B(x,2r) \subseteq J_p, \tag{1}$$

$$J_l \diamond J_p, \quad J_l \subseteq_\forall J_{\varphi(u)}, \quad J_l \subseteq_\forall J_{f(i)} \quad \text{and} \quad J_q \subseteq_\forall J_{f(i)}. \tag{2}$$

J_l intersects S (in x), so $\exists j\big(f(j) = l\big)$. (1) $\Rightarrow S \subseteq J_p \cup J_q \cup J_{f(j)}$ and so $\Omega\big(p,q,f(j)\big)$. Also $J_{f(j)} \subseteq_\exists J_{h(k)}$ and so $\Gamma(i,k,j)$. $\square$

Selection theorem and primitive recursion

By selection theorem, there is a recursive $\gamma\colon \mathbb{N}^2 \to \mathbb{N}$ such that $\Gamma\big(i, k, \gamma(i, k)\big)$ for all $i, k \in \mathbb{N}$. Define by primitive recursion

$$\begin{aligned} \psi(i, 0) &:= \gamma(i, 0), \\ \psi(i, k+1) &:= \gamma\big(\psi(i, k), k+1\big). \end{aligned}$$

Then ψ is a recursive function such that

$$\Gamma\big(\psi(i, k), k+1, \psi(i, k+1)\big).$$

for all $i, k \in \mathbb{N}$. Let $i \in \mathbb{N}$. Then for each $k \in \mathbb{N}$ we have

$$\begin{aligned} (Claim1): &\qquad \overline{J_{f(\psi(i,k+1))}} \cap S \subseteq J_{f(\psi(i,k))} (\cap S), \\ (\Gamma 2): &\qquad J_{f(\psi(i,k))} \subseteq_\exists J_{h(k)}. \end{aligned}$$

Diameters tend to 0

Using the claims at the bottom of previous slide,

$$\mathrm{mesh}_k := \max_{v\in[h(k)]} \mathrm{diam}(I_v \cap S)$$

$$U_k := J_{f(\psi(i,k))} \cap S \subseteq I_u \cap S \text{ for some } u \in [h(k)]$$

$$d_k := \mathrm{diam}\, U_k \leq \mathrm{diam}(I_u \cap S) \leq \mathrm{mesh}_k$$

$$U_{k+1} \subseteq \overline{U_{k+1}} \subseteq \overline{J_{f(\psi(i,k))}} \cap \overline{S} \subseteq U_k, \text{ so } (d_k)_k \text{ is decreasing}$$

S metric & compact $\Rightarrow$ totally bounded $\Rightarrow$

covered by $J_{h(k)}$ with arbitrarily small cells

$$\Longrightarrow (\forall \varepsilon > 0)\, \exists k\, (\mathrm{mesh}_k < \varepsilon)$$

Conclusion: $\lim_k d_k = 0$ (for every i)

Constructing the sequence

Also, diam $\overline{U_k} = \text{diam } U_k = d_k \xrightarrow{k} 0$
(and they're all nonempty by definition of f), so
by Cantor's intersection theorem, $\exists!\, x_i \in \bigcap_k \overline{U_k}$.

But also (for every k)

$$x_i \in \overline{U_{k+1}} = \overline{J_{f(\psi(i,k+1))} \cap S} \subseteq \overline{J_{f(\psi(i,k+1))}} \cap \overline{S} = F_{k+1} \subseteq U_k.$$

Particularly, $x_i \in U_0 \subseteq J_{f(\psi(i,0))} = J_{f(\gamma(i,0))} \subseteq_\forall J_{f(i)}$ (by (Γ1)).

But among $J_{f(i)}$ are all base open sets intersecting S.
Therefore $(x_i)_i$ is dense in S.

Computability

For computability of $(x_i)_i$ it is enough to prove

$$x_i \in I_j \iff \exists k \big(J_{f(\psi(i,k))} \subsetneq_\forall J_{\varphi(j)} \big)$$

(because the right hand side is r.e.).
($\Leftarrow$) is easy: for that k, we have $x_i \in U_k \subseteq J_{f(\psi(i,k))} \subseteq J_{\varphi(j)} = I_j$.

($\Rightarrow$) Assume $x_i \in I_j$. We seek k such that $j \in [h(k)]$ and $x_i \notin I_u$ for every $u \in [h(k)] \setminus \{j\}$: then the right hand side will follow by (Γ2).

Since $x_i \in I_j \cap S$, there is an $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq I_j \cap S$.
By compactness of $S \setminus B(x, \varepsilon)$, it is covered by finitely many Is which avoid x.
Add I_j among them and we get J_w for $w \in \operatorname{rng} h$ (since we have covered the whole S), and the only I containing x is precisely I_j.