A good method of transforming Veltman into Verbrugge models

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Contents

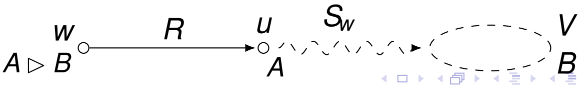
Introduction: Interpretability logic IL

We will assume that you're familiar with the following concepts:

- interpretability logic $\mathtt{IL}$, Veltman frames and Veltman models
- generalised Veltman semantics - nowadays it is called **Verbrugge semantics** in honor of Rineke Verbrugge
- A ~~generalised Veltman~~ **Verbrugge model** is a quadruple $\mathfrak{M} = \big(W, R, \{S_w \mid w \in W\}, \Vdash\big)$, where
  - the first three components form a ~~generalised Veltman~~ **Verbrugge frame**,
  - V is a valuation mapping propositional variables to subsets of W.

The forcing relation $\mathfrak{M}, w \Vdash A$ is defined as in definition of Veltman models with the difference that now

$$\mathfrak{M}, w \Vdash A \rhd B \quad \Leftrightarrow \quad \forall u\Big(wRu \mathbin{\&} u \Vdash A \Rightarrow \exists V(uS_wV \mathbin{\&} V \Vdash B)\Big).$$



Introduction: Bisimulations and bisimulation games

- basic equivalence between Veltman models: bisimulations
- M. Vuković defined bisimulations (and their finite approximations called *n*-bisimulations) for Verbrugge semantics
- M. Vuković and D. Vrgoč proved: *n*-bisimilar worlds are *n*-modally equivalent
- converse is generally not true, not even with finite set of propositional variables
- that lead us to new notions of bisimulations for Verbrugge semantics called w-bisimulations and their corresponding games called weak bisimulation games
- why games are important:
  - A. Dawar and M. Otto developed a *models-for-games method*, which provides conditions from which a Van Benthem characterisation theorem over a particular class of models immediately follows
  - using bisimulation games on Veltman models for interpretability logic, M. Vuković and T. Perkov proved that this result can be extended to Veltman models for the interpretability logic IL

A **w-bisimulation** between two Verbrugge models $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ and $\mathfrak{M}' = (W', R', \{S'_{w'} : w' \in W'\}, \Vdash)$ is a nonempty binary relation $Z \subseteq W \times W'$ such that the following conditions hold:

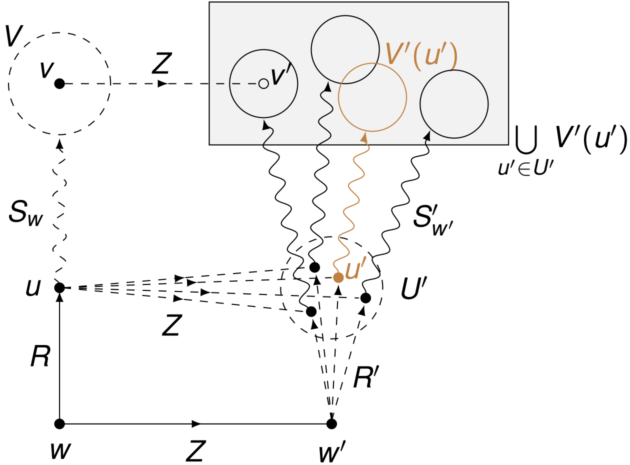
(at) *If wZw' then $w \Vdash p$ if and only if $w' \Vdash p$, for all propositional letters p;*

(w-forth) *If wZw' and wRu, then there exists a nonempty set $U' \subseteq W'$ such that for all $u' \in U'$, uZu' and $w'R'u'$, and for each function $V' : U' \to \mathcal{P}(W')$ such that for all $u' \in U'$, $u'S'_{w'}V'(u')$, there exists set V with uS_wV and for all $v \in V$ there exists $v' \in \bigcup_{u' \in U'} V'(u')$ with vZv';*

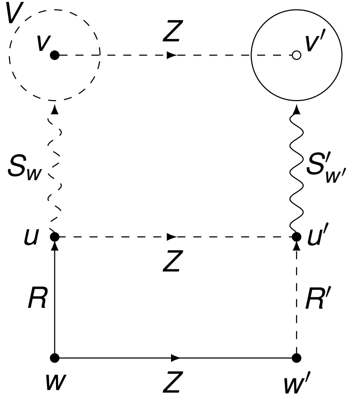
(w-back) *If wZw' and $w'R'u'$, then there exists a nonempty set $U \subseteq W$ such that for all $u \in U$, uZu' and wRu, and for each function $V : U \to \mathcal{P}(W)$ such that for all $u \in U$, $uS_wV(u)$, there exists set V' with $u'S'_{w'}V'$ and for all $v' \in V'$ there exists $v \in \bigcup_{u \in U} V(u)$ with vZv'.*

When Z is a w-bisimulation linking the nodes $w \in W$ and $w' \in W'$ we say that w and w' are **w-bisimilar**. Notation: $w \leftrightsquigarrow w'$ (for bisimulations, the sign $\leftrightarrow$ is used).

Illustration of (w-forth) condition (compared to the (forth) condition)



The w-forth condition...



... and (forth) condition from the definition of bisimulation

Finite w-bisimulations

An ***n*-w-bisimulation** between two Verbrugge models $\mathfrak{M} = (W, R, S, \Vdash)$ and $\mathfrak{M}' = (W', R', S', \Vdash')$ is a decreasing sequence of relations

$$Z_n \subseteq Z_{n-1} \subseteq \cdots \subseteq Z_1 \subseteq Z_0 \subseteq W \times W'$$

that possesses the following properties:

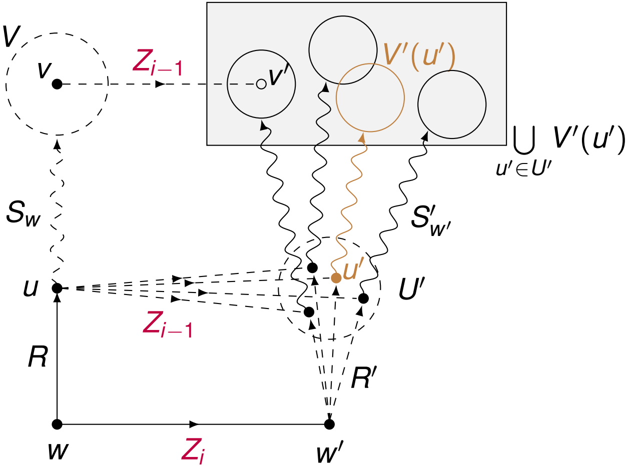
(at) *If $w \boxed{Z_0} w'$ then $w \Vdash p$ if and only if $w' \Vdash p'$, for all prop. letters p;*

(n-w-forth) *For every i from 1 to n, if $w \boxed{Z_i} w'$ and wRu then there exists a nonempty set $U' \subseteq W'$ such that for all $u' \in U'$, $u \boxed{Z_{i-1}} u'$ and $w'R'u'$, and for each function $V' : U' \to \mathcal{P}(W')$ such that for all $u' \in U'$, $u'S'_{w'}V'(u')$, there exists set V with uS_wV and for all $v \in V$ there exists $v' \in \bigcup_{u' \in U'} V'(u')$ with $v \boxed{Z_{i-1}} v'$;*

(n-w-back) *similar to (n-w-forth) (with roles of $\mathfrak{M}$ and $\mathfrak{M}'$ interchanged).*

When $Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_n$ is an n-w-bisimulation linking two nodes $w \in W$ and $w' \in W'$ we say that w and w' are ***n*-w-bisimilar**.

Illustration of n-w-forth condition

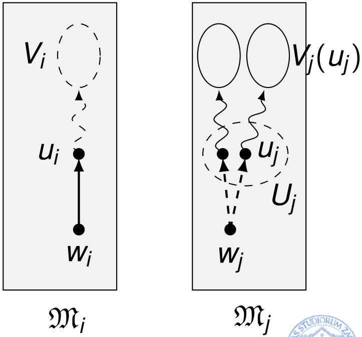


The n-w-forth condition.

- Let $\mathfrak{M}_0 = (W_0, R_0, \{S_w^{(0)} : w \in W_0\}, \Vdash)$ and $\mathfrak{M}_1 = (W_1, R_1, \{S_w^{(1)} : w \in W_1\}, \Vdash)$ be two Verbrugge models.
- The **w-bisimulation game** is played by two players, *Challenger* and *Defender*, who move from one configuration to the other in a series of consecutive rounds.
- A configuration is a tuple $(\mathfrak{M}_0, w_0, \mathfrak{M}_1, w_1)$, where $w_0 \in W_0$ and $w_1 \in W_1$.
- Every round is played from some configuration $(\mathfrak{M}_0, w_0, \mathfrak{M}_1, w_1)$. At the beginning of each round, it is checked that w_0 and w_1 satisfy the same propositional variables. If that check fails, the Challenger wins and game is over.

A single round, starting with configuration $(\mathfrak{M}_0, w_0, \mathfrak{M}_1, w_1)$, is played as follows:

1. Challenger chooses $i \in \{0, 1\}$, index of one Verbrugge model.
   We denote $j := 1 - i$, the index of another model.
2. Challenger picks $u_i \in W_i$ such that $w_i R_i u_i$.
   If there are no such worlds,
   Defender wins and game is over.
3. Defender picks $U_j \subseteq W_j$ such that $(\forall u_j \in U_j)(w_j R_j u_j)$.
   If there are no such sets U_j,
   Challenger wins and game is over.
4. Challenger picks some function $V_j : U_j \to \mathcal{P}(W_j)$
   such that $(\forall u_j \in U_j)(u_j S^{(j)}_{w_j} V_j(u_j))$.
5. Defender picks some $V_i \subseteq W_i$ such that $u_i S^{(i)}_{w_i} V_i$.



How to select the starting configuration for the next round

The configuration $(\mathfrak{M}_0, w, \mathfrak{M}_1, w')$ from which the next round starts is selected as follows:

(i) Challenger picks some world $u_j \in U_j$ or some world $v_i \in V_i$.

(ii) If $u_j \in U_j$ was picked, the next round is played from the configuration $(\mathfrak{M}_0, u_0, \mathfrak{M}_1, u_1)$. If $v_i \in V_i$ was picked, then Defender picks some world $v_j \in \bigcup_{u_j \in U_j} V_j(u_j)$ and the next round is played from the configuration $(\mathfrak{M}_0, v_0, \mathfrak{M}_1, v_1)$.

An ***n*-w-bisimulation game** is a w-bisimulation game that ends after n rounds. If Challenger did not win in the n-w-bisimulation game, then by definition we consider Defender to have won.

Winning strategies in a n-w-game and n-w-bisimulations

Proposition

Let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ and $\mathfrak{M}' = (W', R', \{S'_w : w \in W'\}, \Vdash)$ be two Verbrugge models and $w \in W$, $w' \in W'$ be worlds in them. For each $n \in \mathbb{N}$, Defender has a winning strategy in an n-w-game with a starting configuration $(\mathfrak{M}, w, \mathfrak{M}', w')$ if and only if w and w' are n-w-bisimilar.

- for $\boxed{\Rightarrow}$ direction, we define (for k from 0 to n)

$$Z_k := \{(v, v') \in W \times W' : \text{Defender has a winning strategy in an } k\text{-w-game starting with } (\mathfrak{M}, v, \mathfrak{M}', v')\}.$$

- for $\boxed{\Leftarrow}$ direction, Defender can use the n-w-bisimulation to pick out elements in his winning strategy

n-modal equivalence implies n-w-bisimilarity...

...with finite set of propositional variables!

It can be proved that if $Z \subseteq W \times W'$ is a (n-)bisimulation, then Z is also a (n-)w-bisimulation (and that the converse doesn't hold). Also, now we get:

Theorem

Assume that the set of propositional variables is finite and let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ and $\mathfrak{M}' = (W', R', \{S'_w : w \in W'\}, \Vdash)$ be two Verbrugge models. Let $n \in \mathbb{N}$, $w \in W$ and $w' \in W'$. If w and w' are n-modally equivalent then they are n-w-bisimilar.

- proof by induction on n
- the interesting part is the induction step $(n+1)$ where we define a winning strategy for the Defender in the $(n+1)$-w-bisimulation game starting with the configuration $(\mathfrak{M}, w, \mathfrak{M}', w')$

Modal equivalence and w-bisimulation

It can be shown by an easy induction that w-bisimiliraty implies modal equivalence.

Proposition

Let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ and $\mathfrak{M}' = (W', R', \{S'_w : w \in W'\}, \Vdash)$ be two Verbrugge models and $w \in W$, $w' \in W'$ two worlds in them.

(a) If $\mathfrak{M}_0, w_0 \underline{\leftrightsquigarrow}_n \mathfrak{M}_1, w_1$ then $\mathfrak{M}_0, w_0 \equiv_n \mathfrak{M}_1, w_1$.

(b) If $\mathfrak{M}_0, w_0 \underline{\leftrightsquigarrow} \mathfrak{M}_1, w_1$ then $\mathfrak{M}_0, w_0 \equiv \mathfrak{M}_1, w_1$.

The main question now is does the converse hold.

> *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two Verbrugge models and $w \in W$, $w' \in W'$ two worlds in them. If $w \equiv w'$, does then $w \underline{\leftrightsquigarrow} w'$ hold?*

We will prove that the answer to that is **no** by using a modified procedure that was used by V. Čačić and D. Vrgoč in the case of Veltman models.

Modal equivalence does not imply bisimilarity

- a standard result for **Kripke models** from:
  P. Blackburn, M. de Rijke, Y. Venema, *Modal Logic*, Cambridge Univ. Press, 2001.

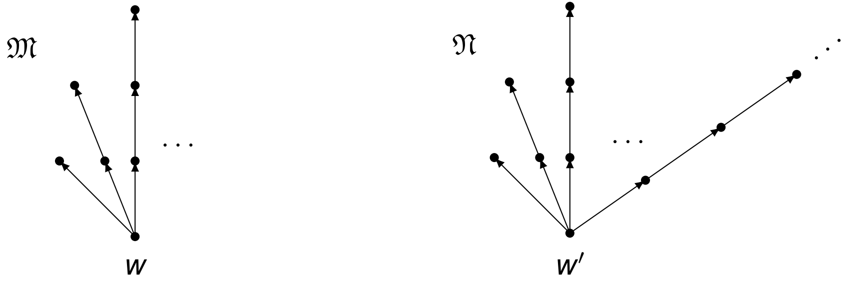


Figure: w and w' are modally equivalent but not bisimilar

Modal equivalence does not imply bisimilarity

- result for **Veltman models** from:
  V. Čačić, D. Vrgoč, *A Note on Bisimulation and Modal Equivalence in Provability Logic and Interpretability Logic*, Studia Logica 101(2013), 31–44

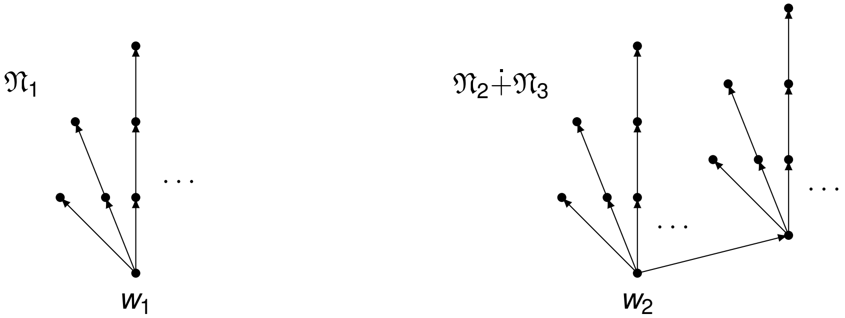


Figure: w and w' are modally equivalent but not bisimilar

A method for obtaining Veltman models from GL-models

- Let $\mathfrak{N} = (W, R, V)$ be a **GL**-model. For every $w \in W$ we define

$$uS_w v \quad \text{if and only if} \quad wRu\underline{R}v,$$

where we denote the reflexive closure of R with $\underline{R}$. We denote $(W, R, \{S_w : w \in W\}, V)$ by $Vel\ \mathfrak{N}$.

Theorem

The worlds w_1 and w_2, in Veltman models $\mathfrak{M}_1 \equiv Vel\ \mathfrak{N}_1$ and $\mathfrak{M}_2 \equiv Vel\ (\mathfrak{N}_1 \dot{+} \mathfrak{N}_2)$, are modally equivalent but not bisimilar.

Definition

Let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ be a Veltman model. For every $w \in W$ and $V \subseteq R[w]$ we define

$$v\overline{S}_w V \quad :\Longleftrightarrow \quad (\exists u \in V)(vS_w u).$$

We denote $(W, R, \{\overline{S}_w : w \in W\}, \Vdash)$ by $Ver\ \mathfrak{M}$.

- It is easy to check that $Ver\ \mathfrak{M}$ is a Verbrugge model.
- It remains to show that the above transformation preserves modal equivalence and (in a way) bisimulations.
- That would give us that the worlds w_1 and w_2, in Verbrugge models $Ver\ \mathfrak{M}_1$ and $Ver\ \mathfrak{M}_2$ are modally equivalent but not w-bisimilar.

Transformation preserves modal equivalence

Theorem

Let F be a IL-formula, $\mathfrak{M} = (W, R, \{S'_w : w \in W\}, \Vdash)$ Veltman model and $Ver\,\mathfrak{F} = (W, R, \{S_w : w \in W\}, \Vdash)$. Then for every world $w \in W$:

$$\mathfrak{M}, w \Vdash F \quad \text{if and only if} \quad Ver\,\mathfrak{M}, w \Vdash' F.$$

- proof: by induction on the complexity of the formula F

Transformation preserves (in a way) bisimulations

Proposition

Let $\mathfrak{M} = \big(W, R, \{S_w : w \in W\}, \Vdash\big)$ and $\mathfrak{M}' = \big(W', R', \{S'_w : w \in W'\}, \Vdash\big)$ be two Veltman models, $w_0 \in W$, $w'_0 \in W'$ two worlds, and $Ver\ \mathfrak{M} = \big(W, R, \{\overline{S}_w : w \in W\}, \Vdash\big)$, $Ver\ \mathfrak{M}' = \big(W', R', \{\overline{S}'_w : w \in W'\}, \Vdash\big)$ Verbrugge models. Then:

$$Ver\ \mathfrak{M}, w_0 \mathrel{\underline{\leftrightsquigarrow}} Ver\ \mathfrak{M}', w'_0 \quad \text{if and only if} \quad \mathfrak{M}, w_0 \mathrel{\underline{\leftrightarrow}} \mathfrak{M}', w'_0.$$

Proof.

$\boxed{\Leftarrow}$ This direction follow directly from the following two facts: similar result exists for bisimulations, and bisimulation implies w-bisimulations. Now we have:

- $\mathfrak{M}, w_0 \mathrel{\underline{\leftrightarrow}} \mathfrak{M}', w'_0 \quad \Rightarrow \quad Ver\ \mathfrak{M}, w_0 \mathrel{\underline{\leftrightarrow}} Ver\ \mathfrak{M}', w'_0$
- $Ver\ \mathfrak{M}, w_0 \mathrel{\underline{\leftrightarrow}} Ver\ \mathfrak{M}', w'_0 \quad \Rightarrow \quad Ver\ \mathfrak{M}, w_0 \mathrel{\underline{\leftrightsquigarrow}} Ver\ \mathfrak{M}', w'_0$

Transformation preserves (in a way) bisimulations

$\Rightarrow$ We need to prove:

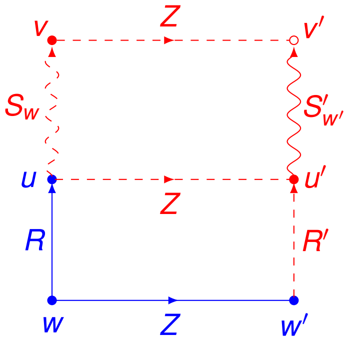
$$\text{if} \quad Ver\,\mathfrak{M}, w_0 \underline{\leftrightsquigarrow} Ver\,\mathfrak{M}', w_0' \quad \text{then} \quad \mathfrak{M}, w_0 \leftrightarrow \mathfrak{M}', w_0'$$

- note that this is the important direction (the contraposition of this statement will be used to get our result regarding w-bisimulation and modal equivalence)
- assume $Ver\,\mathfrak{M}, w_0 \underline{\leftrightsquigarrow} Ver\,\mathfrak{M}', w_0'$
- denote by Z a w-bisimulation such that $(w_0, w_0') \in Z$
- by definition of w-bisimulation, Z satisfies (at), (w-forth) and (w-back) conditions
- in order to show that $\mathfrak{M}, w_0 \leftrightarrow \mathfrak{M}', w_0'$, it suffices to prove that Z satisfies the (forth) condition from definition of bisimulation of Veltman models (the (back) condition can be proven analogously)

Transformation preserves (in a way) bisimulations

- assume wZw' and wRu - we need to show that there exists $u' \in W'$ such that:

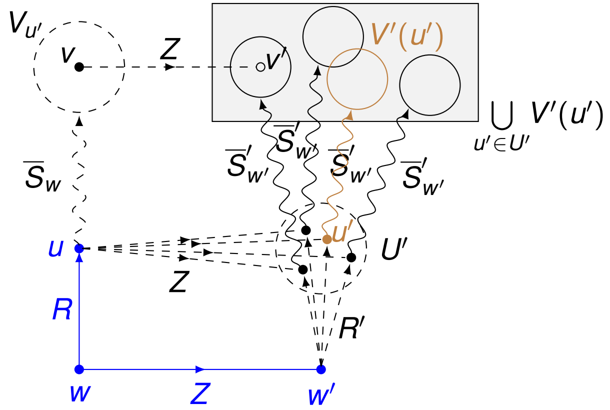
$$uZu' \text{ i } w'R'u' \text{ i } (\forall v' \in W')(u'S'_{w'}v' \Rightarrow (\exists v \in W)(uS_w v \text{ i } vZv'))$$



What we have by (w-forth) condition:

there exists non-empty set $U' \subseteq W'$ such that $(\forall u' \in U')(uZu'$ and $w'R'u')$ and for each function $V' : U' \to \mathcal{P}(W')$ such that for all $u' \in U'$, $u'\overline{S}'_{w'}V'(u')$,

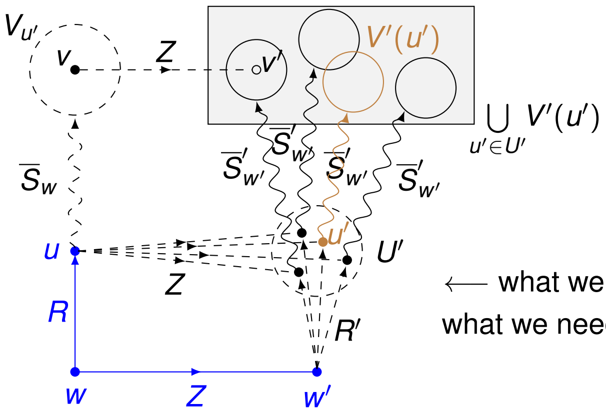
$$(\exists V_{u'} \subseteq W)\Big(u\overline{S}_w V_{u'} \text{ and } (\forall v \in V_{u'})(\exists v' \in \bigcup_{u' \in U'} V'(u'))(vZv')\Big).$$



Transformation preserves (in a way) bisimulations

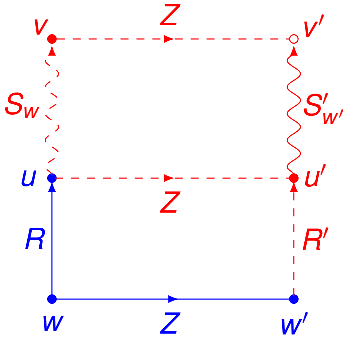
Now we can see that is suffices to choose some $u' \in U'$ such that:

$$(\forall v' \in W')\Big(u' S'_{w'} v' \Rightarrow (\exists v \in W)(u S_w v \;\text{ i }\; v Z v')\Big).$$



$\longleftarrow$ what we have

what we need $\longrightarrow$

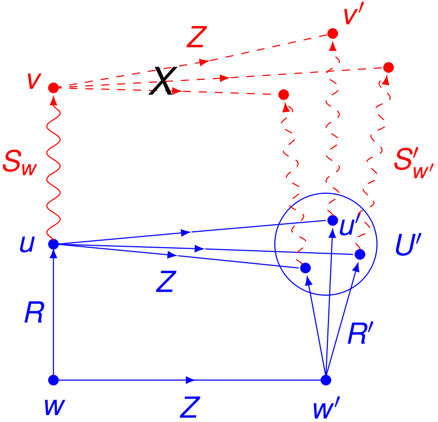


Problem: we don't know which $u' \in U'$ to choose!

Let's assume the opposite:

there is no $u' \in U'$ with the required property, i.e.

$$(\forall u' \in U')(\exists v' \in W')\Big(u' S'_{w'} v' \text{ i } (\forall v \in W)(u S_w v \Rightarrow \neg(vZv'))\Big).$$



$\Rightarrow$ for every $u' \in U'$ we can choose one $v'_{u'} \in W'$ such that

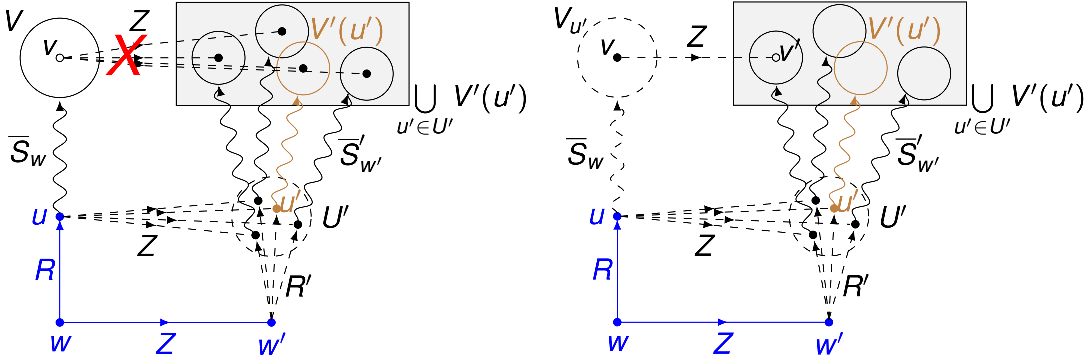
- $u' S'_{w'} v'_{u'}$
- $(\forall v \in W)(u S_w v \Rightarrow v Z v'_{u'})$

$\Rightarrow$ we can define a function $V' : U' \to \mathcal{P}(W')$,

$$V'(u') = \{v'_{u'}\}, \quad \forall u' \in U'$$

- Note: by definition of $\overline{S}'_{w'}$, $u \overline{S}'_{w'} V'(u')$.

The rest of the proof is shown in the following pictures:



We have the situation shown on the left, so we get a contradiction with properties of U' by the (w-forth) property of Z (shown on the right). ∎

w-bisimilarity does not imply modal equivalence

Now we have all the tools that we need in order to prove:

Theorem

Worlds w_1 and w_2 in Verbrugge models $Ver\,\mathfrak{M}_1$ and $Ver\,\mathfrak{M}_2$ are modally equivalent, but not w-bisimilar.

Proof.

- we already now that w_1 and w_2 are modally equivalent and not bisimilar as worlds of Veltman models $\mathfrak{M}_1$ and $\mathfrak{M}_2$
- because our transformation preserves modal equivalence, they are modally equivalent as worlds of Verbrugge models $Ver\,\mathfrak{M}_1$ and $Ver\,\mathfrak{M}_2$
- using the previous proposition, we get that they are not w-bisimilar

The End.

Questions?