The logic ILP for intuitionistic reasoning about probability

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Outline

Previous research

- Study of intuitionistic logic was initiated by Kolmogorov, Glivenko and Heyting
- An idea of extending intuitionistic logic with probabilistic operators is proposed in Boričić, B. and Rašković, M., "A probabilistic validity measure in intuitionistic propositional logic"
- The paper presents a complete axiom system for semantics which consists of intuitionistic Kripke models in which each possible world is equipped with two partial functions representing inner and outer probability measures, such that outer probabilities cannot increase, and inner probabilities cannot decrease.

Previous research

- The papers
  - Marković, Z.,Ognjanović, Z. and Rašković, M., "A probabilistic extension of intuitionistic logic",
  - Marković, Z.,Ognjanović, Z. and Rašković, M., "An intuitionistic logic with probabilistic operators"
  - Marković, Z, Ognjanović, Z. and Rašković, M., "What is the Proper Propositional Base for Probabilistic Logic?",

  give strongly complete axiomatizations for probabilistic logics based on intuitionistic propositional calculus and provide some real-life examples illustrating that such approach might be preferable to probabilistic logic based on classical logic.
- Those papers start with intuitionistic calculus, but use classical reasoning about probabilistic formulas.

Contribution of this research

We continue along the lines of these papers and describe a propositional logic in which:

- Finitely additive (and not inner and outer) probabilities are considered, while the corresponding range is the unit interval of rational numbers.
- Probabilistic operators behave in accordance with intuitionistic laws.

Contribution of this research

- We consider formulas of the form $P_{\geqslant r}\alpha$ and $P_{\leqslant r}\alpha$ as atomic formulas. Their truthfulness is understood as "it is effectively provable that the probability of α is at least (at most) r", while $\neg P_{\geqslant r}\alpha$ is read "it is effectively refutable that the probability of α is at least r"
- $P_{\geqslant 0}\alpha$ ($\neg P_{\geqslant 0}\alpha$) means that α is measurable (not measurable).
- The corresponding semantics consist of Kripke models with partially ordered possible worlds. Any possible world w can be seen as a description of uncertain knowledge.
- This allows us to imagine a possible world w such that there is a statement α such that neither α nor $\neg\alpha$ are forced in w. It might be that one of those statements can be forced afterwards, but once established truthfulness of statements cannot be changed in later worlds.
- This is in accordance with the intuition that intuitionistic logic may be viewed as the logic of the growth of knowledge.

Contribution of this research

- Let the formula β means "an accident was caused by a drunk driver"
- Then $P_{\geqslant \frac{2}{3}}\beta$ means "The probability of β is at least $\frac{2}{3}$", while $\neg P_{\geqslant \frac{2}{3}}\beta$ says "The probability of β is not greater or equal to $\frac{2}{3}$".
- According to available evidences, it may happen that: it is not proved that the probability that an accident was caused by a drunk driver is at least $\frac{2}{3}$, and it is not proved that the probability that an accident was caused by a drunk driver is not at least $\frac{2}{3}$.
- Thus, neither $P_{\geqslant \frac{2}{3}}\beta$ nor $\neg P_{\geqslant \frac{2}{3}}\beta$ is forced, which reminds us of the well known fact that the principle of excluded middle is not intutionistically valid.

Contribution of this research

- Compactness (“every finitely satisfiable set of formulas is satisfiable”) does not hold for our logic.
- we present an infinitary logic and prove that it is strongly complete (“a formula is a syntactical consequence of a set of formulas iff it is a semantical consequence of the set”) and decidable.

Syntax and semantics

The formal language of *ILP* consists of a nonempty at most countable set of propositional letters denoted *Var* and the following operators:

- classical: $\neg$, $\wedge$,
- $P_{*r}\alpha$ $* \in \{\geqslant, \leqslant, >, <, =\}$ and $r \in [0,1]_{\mathbb{Q}}$.

Syntax and semantics

- The set of classical formulas For_C is inductively defined in the following way:
$$p \mid \neg\alpha \mid \alpha \wedge \beta \mid \alpha \vee \beta.$$
- The set of probabilistic formulas For_P is inductively defined as follows:
$$P_{*r}\alpha \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \to \psi.$$

Syntax and semantics

Definition

A model is a structure
$\mathbb{M} = \langle W, \{H_w : w \in W\}, \{\mu_w \; : \; w \in W\}, \leqslant \rangle$ with the following properties:

- W is a non-empty set of possible worlds,
- $\langle W, \leqslant \rangle$ is a partially ordered set (poset) called a frame;

Syntax and semantics

Definition

- every H_w is a subset of For_C which satisfies:
  - $\bot \in H_w$, $\top \in H_w$;
  - If $\alpha \in H_w$ and α is equivalent to β (i.e. $\alpha \leftrightarrow \beta$ is a classical tautology), then $\beta \in H_w$;
  - $\alpha \in H_w$ iff $\neg\alpha \in H_w$;
- For all $w_1, w_2 \in W$, if $w_1 \leqslant w_2$, then $H_{w_1} \subseteq H_{w_2}$;
- every μ_w is a mapping from H_w to $[0,1]_{\mathbb{Q}}$ which satisfies:
  - $\mu_w(\bot) = 0$, $\mu_w(\top) = 1$;
  - If $w_1 \leqslant w_2$, then μ_{w_1} is a restriction of μ_{w_2} on H_{w_1};
  - If α and β are disjoint (i.e. contradict each other) and if $\alpha, \beta, \alpha \vee \beta \in H_w$, then $\mu_w(\alpha \vee \beta) = \mu_w(\alpha) + \mu_w(\beta)$;
  - If α and β are equivalent and $\alpha, \beta \in H_w$, then $\mu_w(\alpha) = \mu_w(\beta)$. □

Syntax and semantics

Definition

(Forcing relation) Let $\mathbb{M} = \langle W, \{H_w : w \in W\}, \{\mu_w \ : \ w \in W\}, \leqslant \rangle$ be a model and $w \in W$. The forcing relation $\Vdash_{\mathbb{M}}$ between possible worlds and For_P-formulas satisfies:

- $w \Vdash_{\mathbb{M}} P_{\geqslant r}\alpha$ iff $\alpha \in H_w$ and $\mu_w(\alpha) \geqslant r$;
- $w \Vdash_{\mathbb{M}} P_{\leqslant r}\alpha$ iff $\alpha \in H_w$ and $\mu_w(\alpha) \leqslant r$;
- $w \Vdash_{\mathbb{M}} P_{> r}\alpha$ iff $\alpha \in H_w$ and $\mu_w(\alpha) > r$;
- $w \Vdash_{\mathbb{M}} P_{< r}\alpha$ iff $\alpha \in H_w$ and $\mu_w(\alpha) < r$;
- $w \Vdash_{\mathbb{M}} P_{= r}\alpha$ iff $\alpha \in H_w$ and $\mu_w(\alpha) = r$;
- $w \Vdash_{\mathbb{M}} \neg\phi$ iff, for all $v \geqslant w$, $v \nVdash_{\mathbb{M}} \phi$;
- $w \Vdash_{\mathbb{M}} \phi \wedge \psi$ iff $w \Vdash_{\mathbb{M}} \phi$ and $w \Vdash_{\mathbb{M}} \psi$;

Syntax and semantics

Definition

(Forcing relation)

- $w \Vdash_{\mathbb{M}} \phi \vee \psi$ iff $w \Vdash_{\mathbb{M}} \phi$ or $w \Vdash_{\mathbb{M}} \psi$;
- $w \Vdash_{\mathbb{M}} \phi \rightarrow \psi$ iff, for all $v \geqslant w$, either $v \nVdash_{\mathbb{M}} \phi$, or $v \Vdash_{\mathbb{M}} \phi \wedge \psi$.

$\square$

Syntax and semantics

Theorem

Suppose that an ILP–formula ϕ is an instance of some intuitionistic tautology. Then, ϕ is ILP–valid. $\square$

Comments on syntax and semantics and terminology

The above definition of the forcing relation suggests that some probabilistic operators are definable:

- $P_{\leqslant r}\alpha = P_{\geqslant 1-r}\neg\alpha$,
- $P_{>r}\alpha = P_{\geqslant r}\alpha \wedge \neg P_{\leqslant r}\alpha$,
- $P_{<r}\alpha = P_{\leqslant r}\alpha \wedge \neg P_{\geqslant r}\alpha$, and
- $P_{=r}\alpha = P_{\leqslant r}\alpha \wedge P_{\geqslant r}\alpha$,

Non–compactness

Theorem (Non–compactness)

Finite satisfiability does not imply satisfiability.

Non–compactness

Proof.

Let $p \in Var$. We define the theory T by

$$T = \{P_{\geqslant 0}p\} \cup \{\neg P_{=s}p \ : \ s \in [0,1]_{\mathbb{Q}}\}.$$

Clearly, T is not satisfiable since it forces that p has an irrational measure. On the other hand, T is finitely satisfiable. Indeed, for each $s \in [0,1]_{\mathbb{Q}}$ we define the model

$$\mathbb{M}_s = \langle \{s\}, [\bot] \cup [\top] \cup [p] \cup [\neg p], \{\mu_s\} \rangle$$

as follows: $\square$

Non–compactness

Proof.

- $[\alpha] = \{\beta \in For_C \ : \ \alpha \models \beta\}$;
- $(\forall \alpha \in [\bot])\mu_s(\alpha) = 0$;
- $(\forall \alpha \in [p])\mu_s(\alpha) = s$;
- $(\forall \alpha \in [\neg p])\mu_s(\alpha) = 1 - s$;
- $(\forall \alpha \in [\top])\mu_s(\alpha) = 1$.

Let Γ be any finite nonempty subset of T. Then, for all s such that $\neg P_{=s}p \notin \Gamma$, we have that $s \Vdash_{\mathbb{M}_s} \Gamma$, i.e., that $s \Vdash_{\mathbb{M}_s} \phi$ for all $\phi \in \Gamma$. $\square$

Strongly complete axiomatization

Our axiom system consists of seven groups of axioms:

- Heyting axioms,
- Norm axioms,
- Disjunctive closure axiom,
- Bookkeeping axioms,
- Equivalence/Negation axioms
- Monotonicity axioms and
- Additivity axiom,

and two inference rules: modus ponens and an infinitary rule.

Strongly complete axiomatization

Heyting axioms This group of ten axioms establishes the essential proof theoretical properties of the logical connectives $\neg$, $\wedge$, $\vee$ and $\to$.

DedThm1: $\phi \to (\psi \to \phi)$;

DedThm2: $\phi \to (\psi \to \theta) \to ((\phi \to \psi) \to (\phi \to \theta))$;

Con1: $(\phi \wedge \psi) \to \phi$;

Con2: $(\phi \wedge \psi) \to \psi$;

Con3: $\phi \to (\psi \to (\phi \wedge \psi))$;

Dis1: $\phi \to (\phi \vee \psi)$;

Dis2: $\psi \to (\phi \vee \psi)$;

Dis3: $(\phi \to \theta) \to ((\psi \to \theta) \to ((\phi \vee \psi) \to \theta))$;

Neg1: $(\phi \to \neg\psi) \to (\psi \to \neg\phi)$;

Neg2: $\neg\phi \to (\phi \to \psi)$.

Strongly complete axiomatization

Norm axioms This group of two axioms ensures that the range of measures values are within $[0, 1]_{\mathbb{Q}}$, and that all classical propositional tautologies and contradictions are measurable. Tautologies have the maximal measure ($= 1$) while contradictions have the minimal measure ($= 0$).

Bnd1: $P_{\geqslant 0}\alpha \rightarrow \neg P_{<0}\alpha \wedge \neg P_{>1}\alpha$;

Bnd2: $P_{=1}\top \wedge P_{=0}\bot$.

Strongly complete axiomatization

Disjunctive closure axiom This axiom is essential for the construction of the canonical model. It allows us to pinpoint the probability of the measurable formula α corresponding to the particular disjunctive closed extension of $P_{\geqslant 0}\alpha$.

$$\text{DisCl: } P_{\geqslant 0}\alpha \rightarrow (P_{>s}\alpha \vee P_{<s}\alpha \vee P_{=s}\alpha).$$

Strongly complete axiomatization

Bookkeeping axioms This group of eight axioms ensure that operators P_{*r}, $* \in \{\leqslant, \geqslant, >, <, =\}$, preserve the ordering of rational numbers from the real unit interval [0,1].

Ord1: $P_{\geqslant r}\alpha \to \neg P_{<r}\alpha$;

Ord2: $P_{>r}\alpha \to (\neg P_{<r}\alpha \wedge \neg P_{\leqslant r}\alpha \wedge P_{\geqslant r}\alpha)$, $r < 1$;

Ord3: $P_{\leqslant r}\alpha \to \neg P_{>r}\alpha$;

Ord4: $P_{<r}\alpha \to (\neg P_{>r}\alpha \wedge \neg P_{\geqslant r}\alpha \wedge P_{\leqslant r}\alpha)$, $r > 0$;

Ord5: $P_{\geqslant s}\alpha \to P_{>t}\alpha$, $t < s$;

Ord6: $P_{\leqslant r}\alpha \to P_{<s}\alpha$, $s > r$;

Ord7: $P_{=r}\alpha \to P_{\geqslant r}\alpha \wedge P_{\leqslant r}\alpha$;

Ord8: $P_{\geqslant r}\alpha \wedge P_{\leqslant r}\alpha \to P_{=r}\alpha$.

Strongly complete axiomatization

Equivalence/Negation axioms This group of three axioms ensure that the set of measurable formulas is closed under equivalence and negation.

EquivNeg1: $P_{\geqslant 0}\alpha \rightarrow P_{\geqslant 0}\beta$, where α and β are equivalent in the classical sense;

EquivNeg2: $P_{\geqslant 0}\alpha \rightarrow P_{\geqslant 0}(\neg\alpha)$;

EquivNeg3: $P_{\geqslant 0}(\neg\alpha) \rightarrow P_{\geqslant 0}\alpha$.

Strongly complete axiomatization

Monotonicity axioms This group of two axioms ensures that all equivalent measurable formulas have the same measure.

Mon1: $P_{\geqslant r}\alpha \wedge P_{\geqslant 0}\beta \to P_{\geqslant r}\beta$, where α implies β in the classical sense;

Mon2: $P_{\geqslant 0}\alpha \wedge P_{\leqslant r}\beta \to P_{\leqslant r}\alpha$, where α implies β in the classical sense;

Strongly complete axiomatization

Additivity axiom The final axioms provides finite additivity.

Add:
$P_{=r}\alpha \wedge P_{=s}\beta \wedge P_{=0}(\alpha \wedge \beta) \wedge P_{\geqslant 0}(\alpha \vee \beta) \rightarrow P_{=r+s}(\alpha \vee \beta)$,
$r + s \leqslant 1$.

Strongly complete axiomatization

Inference rules

There are two inference rules: modus ponens (MP) and the rational range rule (QRng).

MP: From ϕ and $\phi \rightarrow \psi$ infer ψ;

QRng: From the set of premises

$$\{\phi \rightarrow P_{\geqslant 0}\alpha\} \cup \{\phi \rightarrow (P_{=s}\alpha \rightarrow \psi) \ : \ s \in [0,1]_{\mathbb{Q}}\}$$

infer $\phi \rightarrow \psi$.

Strongly complete axiomatization

Inference rules

If $\phi = \top$ (intuitionistic tautology) and $\psi = \bot$, then by QRng, theory

$$T = \{P_{\geqslant 0}\alpha\} \cup \{\neg P_{=s}\alpha \ : \ s \in [0,1]_{\mathbb{Q}}\}$$

is inconsistent. Hence, QRng syntactically forces the rational unit interval as the range of studied measures.

A deductive (or syntactical) closure of a theory T, denoted by $T_\vdash$, is defined by

$$T_\vdash = \{\phi \ : \ T \vdash \phi\}.$$

Definition

A theory T is disjunctively closed iff it has the following property: if $\phi \vee \psi \in T$, then $\phi \in T$ or $\psi \in T$. □

Definition

A theory T is deductively closed iff $T = T_\vdash$. □

Soundness

Here we shall prove that the introduced infinitary inference rule QRng is sound with respect to the class of introduced models. The cases of axioms and Rule MP are straightforward.

Theorem

Let $\mathbb{M} = \langle W, \{H_w : w \in W\}, \{\mu_w : w \in W\}, \leqslant\rangle$ be any model, $w \in W$, and let $w \Vdash_{\mathbb{M}} \phi \rightarrow P_{\geqslant 0}\alpha$ and $w \Vdash_{\mathbb{M}} \phi \rightarrow (P_{=s}\alpha \rightarrow \psi)$ for all $s \in [0,1]_{\mathbb{Q}}$. Then $w \Vdash_{\mathbb{M}} \phi \rightarrow \psi$.

Strong completeness

Theorem (Deduction theorem)

Let T be an ILP–theory and ϕ, ψ arbitrary ILP–formulas. Then,

$$T \vdash \phi \to \psi \text{ iff } T, \phi \vdash \psi.$$

Lemma

Suppose that $T, \phi \vee \psi \nvdash \chi$ and $T, \phi \vdash \chi$. Then $T, \psi \nvdash \chi$.

Strong completeness

Lemma

Suppose that $T \nvdash \chi$. Then, there is a theory T^* with the following properties:

1. $T \subseteq T^*$;
2. $T^* = T^*_{\vdash}$;
3. $T^* \nvdash \chi$;
4. for all $\phi, \psi \in For_P$, $\phi \vee \psi \in T^*$ iff $\phi \in T^*$ or $\psi \in T^*$.

Strong completeness

Theorem (Model existence theorem)

Let $T \nvdash \chi$. Then, there is a model $\mathbb{M} = \langle W, \ldots \rangle$ and $w \in W$ such that $w \Vdash_{\mathbb{M}} T$ and $w \nVdash_{\mathbb{M}} \chi$.

Theorem (Completeness Theorem)

$T_{\vdash} = T_{\Vdash}$.

Decidability

Theorem

A formula $\phi \in \mathit{For}_P$ is satisfiable iff it is satisfiable in a finite model containing at most $2^{|\phi|^2+1}$ worlds.

Theorem

Satisfiability of For_P–formulas is decidable.