Towards Logic of Combinatory Logic

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1. Syntax
2. Axiomatization
3. Semantics
4. Results: Soundness and Completeness
5. Related work and future work

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$$M : \sigma$$

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$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \sigma} \qquad \frac{}{\Gamma \vdash \mathsf{S} : (\sigma \to (\rho \to \tau)) \to (\sigma \to \rho) \to (\sigma \to \tau)}$$

$$\frac{}{\Gamma \vdash \mathsf{K} : \sigma \to (\tau \to \sigma)} \qquad \frac{}{\Gamma \vdash \mathsf{I} : \sigma \to \sigma}$$

$$\frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

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Syntax of LCL

$$\alpha, \beta := M : \sigma \mid \neg\alpha \mid \alpha \wedge \beta$$

Equational theory $\mathcal{EQ}$

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$$M = M \quad (id) \qquad \mathsf{S}MNL = (ML)(NL) \quad (\mathsf{S})$$

$$\mathsf{K}MN = M \quad (\mathsf{K}) \qquad \mathsf{I}M = M \quad (\mathsf{I})$$

$$\frac{M = N}{N = M}\,(\text{sym}) \qquad \frac{M = N \qquad N = L}{M = L}\,(\text{trans})$$

$$\frac{M = N}{MP = NP}\,(\text{app-l}) \qquad \frac{M = N}{PM = PN}\,(\text{app-r})$$

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Rule: $$\frac{Mx = Nx \qquad x \notin FV(M) \cup FV(N)}{M = N}\,(\text{ext})$$

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Rule: $$\frac{Mx = Nx \qquad x \notin FV(M) \cup FV(N)}{M = N}\,(\mathrm{ext})$$

- Equational theory $\mathcal{EQ}$ + rule (ext) = Equational theory $\mathcal{EQ}^{\eta}$.

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Axiomatization

- Combination of the axiomatic system for classical propositional logic and type assignment system for simply typed combinatory logic

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- Eight axiom schemes:

(Ax 1) $\mathsf{S} : (\sigma \to (\tau \to \rho)) \to ((\sigma \to \tau) \to (\sigma \to \rho))$
(Ax 2) $\mathsf{K} : \sigma \to (\tau \to \sigma)$
(Ax 3) $\mathsf{I} : \sigma \to \sigma$
(Ax 4) $(M : \sigma \to \tau) \Rightarrow ((N : \sigma) \Rightarrow (MN : \tau))$
(Ax 5) $M : \sigma \Rightarrow N : \sigma$, if $M = N$ is provable in $\mathcal{EQ}^\eta$
(Ax 6) $\alpha \Rightarrow (\beta \Rightarrow \alpha)$
(Ax 7) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$
(Ax 8) $(\neg\neg\alpha \Rightarrow \neg\beta) \Rightarrow ((\neg\neg\alpha \Rightarrow \beta) \Rightarrow \neg\alpha)$

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(Ax 7)	$(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$
(Ax 8)	$(\neg\neg\alpha \Rightarrow \neg\beta) \Rightarrow ((\neg\neg\alpha \Rightarrow \beta) \Rightarrow \neg\alpha)$

- Inference rule:

$$\frac{\alpha \Rightarrow \beta \qquad \alpha}{\beta}\ (\mathrm{MP})$$

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Semantics

Applicative structure for LCL

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Applicative structure for *LCL*

$$\mathcal{M} = \langle D, \{A^{\sigma}\}_{\sigma}, \cdot, \mathsf{s}, \mathsf{k}, \mathsf{i} \rangle$$

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$$\mathcal{M} = \langle \mathsf{D}, \{A^\sigma\}_\sigma, \cdot, \mathsf{s}, \mathsf{k}, \mathsf{i} \rangle$$

D is a non-empty set, called *domain*

Semantics

Applicative structure for LCL

$$\mathcal{M} = \langle D, \{A^\sigma\}_\sigma, \cdot, \mathsf{s}, \mathsf{k}, \mathsf{i} \rangle$$

$\{A^\sigma\}_\sigma$ is a family of sets indexed by σ such that $A^\sigma \subseteq D$ for all σ, $\{A^\sigma\}_\sigma$ is a short for $\{A^\sigma\}_{\sigma \in \mathsf{Types}}$

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$$\mathcal{M} = \langle D, \{A^\sigma\}_\sigma, \cdot, \mathsf{s}, \mathsf{k}, \mathsf{i} \rangle$$

$\cdot$ is a binary operation on D

$$\cdot \; : \; D \times D \to D$$

It is extensional: for $d_1, d_2 \in D$,

if $(\forall e \in D)(d_1 \cdot e = d_2 \cdot e)$, then $d_1 = d_2$.

It holds that $\cdot \; : \; A^{\sigma \to \tau} \times A^\sigma \to A^\tau$

Semantics

Applicative structure for *LCL*

$$\mathcal{M} = \langle D, \{A^\sigma\}_\sigma, \cdot, \mathsf{s}, \mathsf{k}, \mathsf{i}\rangle$$

s is an element of the domain D, such that

- for every $\sigma, \tau, \rho \in$ Types,

$$\mathsf{s} \in A^{(\sigma\to(\tau\to\rho))\to((\sigma\to\tau)\to(\sigma\to\rho))}$$

- for every $d, e, f \in D$,

$$((\mathsf{s} \cdot d) \cdot e) \cdot f = (d \cdot f) \cdot (e \cdot f)$$

Applicative structure for LCL

$$\mathcal{M} = \langle D, \{A^\sigma\}_\sigma, \cdot, \mathsf{s}, \mathsf{k}, \mathsf{i}\rangle$$

k is an element of the domain D, such that

- for every $\sigma, \tau \in$ Types,

$$\mathsf{k} \in A^{\sigma\to(\tau\to\sigma)}$$

- for every $d, e \in D$,

$$(\mathsf{k} \cdot d) \cdot e = d$$

Applicative structure for *LCL*

$$\mathcal{M} = \langle D, \{A^\sigma\}_\sigma, \cdot, \mathsf{s}, \mathsf{k}, \mathsf{i} \rangle$$

i is an element of the domain D, such that

- for every $\sigma \in$ Types,

$$\mathsf{i} \in A^{\sigma \to \sigma}$$

- for every $d \in D$,

$$\mathsf{i} \cdot d = d$$

Semantics

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$$\mathcal{M} = \langle D, \{A^\sigma\}_\sigma, \cdot, \mathsf{s}, \mathsf{k}, \mathsf{i} \rangle$$

Semantics

Applicative structure for LCL

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An environment ρ for $\mathcal{M}$ is a map from the set of term variables to the domain of the applicative structure $\mathcal{M}$, $\rho : V \to D$.

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Applicative structure for LCL

$$\mathcal{M} = \langle D, \{A^\sigma\}_\sigma, \cdot, \mathsf{s}, \mathsf{k}, \mathsf{i} \rangle$$

An environment ρ for $\mathcal{M}$ is a map from the set of term variables to the domain of the applicative structure $\mathcal{M}$, $\rho : V \to D$.

A LCL-model is a tuple $\mathcal{M}_\rho = \langle \mathcal{M}, \rho \rangle$, where $\mathcal{M}$ is an applicative structure and ρ is an environment for $\mathcal{M}$.

Kašterović, S., Ghilezan, S., *Kripke semantics and completeness for full simply typed lambda calculus,* Journal of Logic and Computation, Volume 30, issue 8 (2020).

Mitchell, J. C., and E. Moggi, *Kripke-style models for typed lambda calculus*, Annals of Pure and Applied Logic, vol. 51, pp. 99–124, 1991.

The interpretation of a term

- $[\![x]\!]_\rho = \rho(x)$;
- $[\![\mathsf{S}]\!]_\rho = \mathsf{s}$;
- $[\![\mathsf{K}]\!]_\rho = \mathsf{k}$;
- $[\![\mathsf{I}]\!]_\rho = \mathsf{i}$;
- $[\![MN]\!]_\rho = [\![M]\!]_\rho \cdot [\![N]\!]_\rho$.

Satisfiability

- $\mathcal{M}_\rho \models M : \sigma$ if and only if $[\![M]\!]_\rho \in A^\sigma$;
- $\mathcal{M}_\rho \models \alpha \wedge \beta$ if and only if $\mathcal{M}_\rho \models \alpha$ and $\mathcal{M}_\rho \models \beta$;
- $\mathcal{M}_\rho \models \neg\alpha$ if and only if it is not true that $\mathcal{M}_\rho \models \alpha$.

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Soundness of $\mathcal{EQ}^\eta$

If $M = N$ in provable in $\mathcal{EQ}^\eta$, then $[\![M]\!]_\rho = [\![N]\!]_\rho$ for any LCL-model $\mathcal{M}_\rho = \langle \mathcal{M}, \rho \rangle$.

Soundness and completeness of equational theory $\mathcal{EQ}^\eta$

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If $M = N$ in provable in $\mathcal{EQ}^\eta$, then $[\![M]\!]_\rho = [\![N]\!]_\rho$ for any LCL-model $\mathcal{M}_\rho = \langle \mathcal{M}, \rho \rangle$.

Completeness of $\mathcal{EQ}^\eta$

If $[\![M]\!]_\rho = [\![N]\!]_\rho$ in every LCL-model, then $M = N$ is provable in $\mathcal{EQ}^\eta$.

Soundness and completeness of the axiomatiation for *LCL*

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Soundness of Ax

If $T \vdash \alpha$, then $T \models \alpha$.

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Related work

S. Feferman, “Constructive theories of functions and classes,” in Proceedings Logic Colloquium ’78, v. D. Boffa, D. and K. McAloon, Eds., vol. 97. Amsterdam: North-Holland, 1979, pp. 159–224, mons, Aug. 24–Sept. 1., 1978.

M. Beeson, “Lambda logic,” in Automated Reasoning - Second International Joint Conference, IJCAR 2004, Cork, Ireland, July 4-8, 2004, Proceedings, ser. Lecture Notes in Computer Science, D. A. Basin and M. Rusinowitch, Eds., vol. 3097. Springer, 2004, pp. 460–474. [Online]. Available: https://doi.org/10.1007/978-3-540-25984-8 34

H. Barendregt, M. W. Bunder, and W. Dekkers, “Systems of illative combinatory logic complete for first-order propositional and predicate calculus,” J. Symb. Log., vol. 58, no. 3, pp. 769–788, 1993. [Online]. Available: https://doi.org/10.2307/2275096

W. Dekkers, M. W. Bunder, and H. Barendregt, “Completeness of two systems of illative combinatory logic for first-order propositional and predicate calculus,” Arch. Math. Log., vol. 37, no. 5-6, pp. 327–341, 1998. [Online]. Available: https://doi.org/10.1007/s001530050102

Motivation and future work

LPP_2

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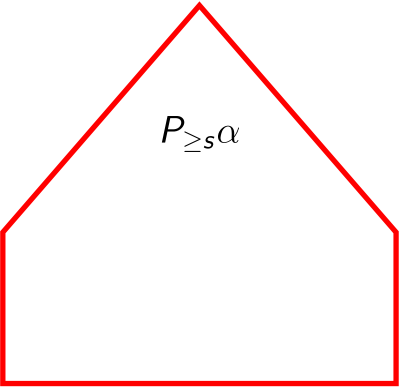

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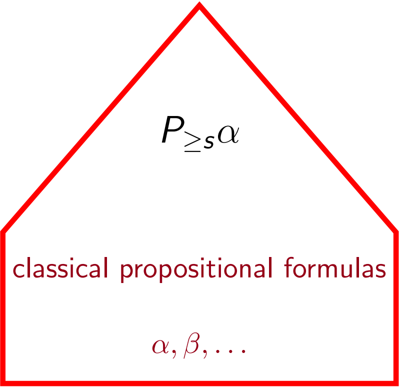

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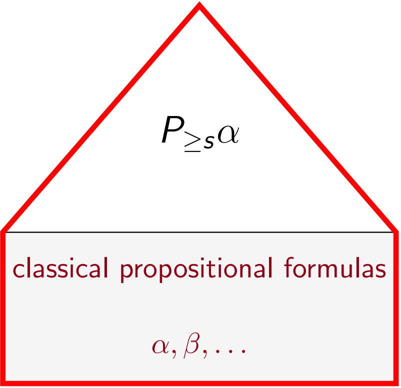

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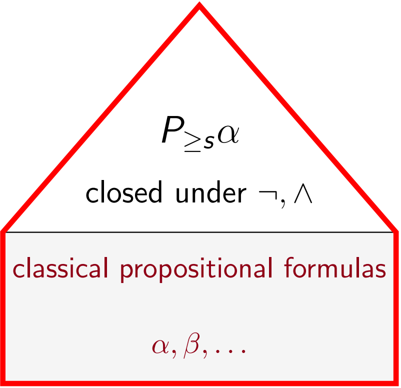

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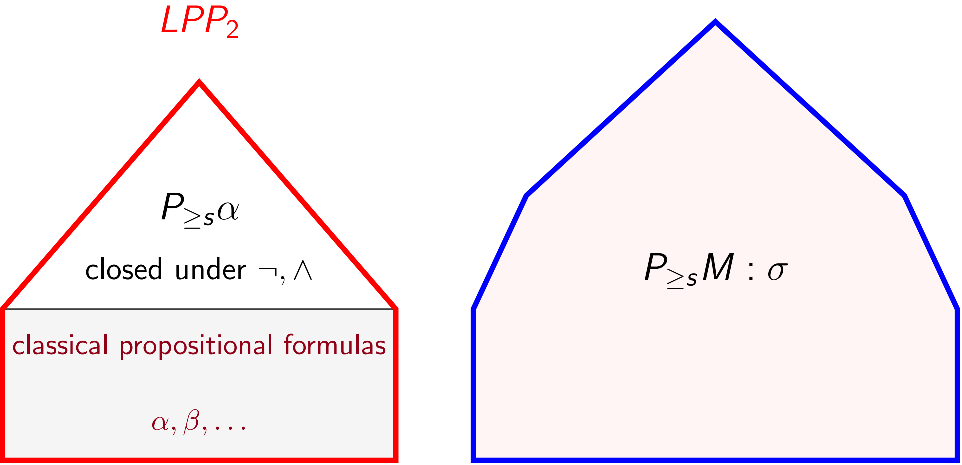

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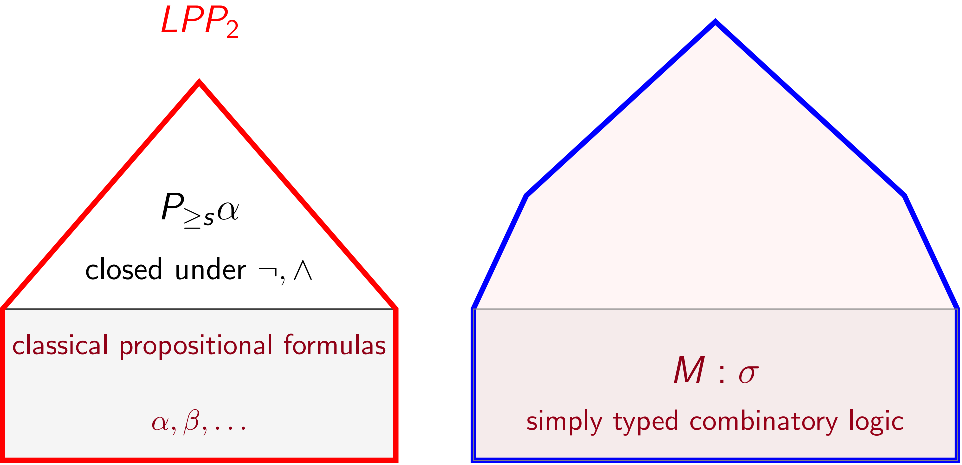

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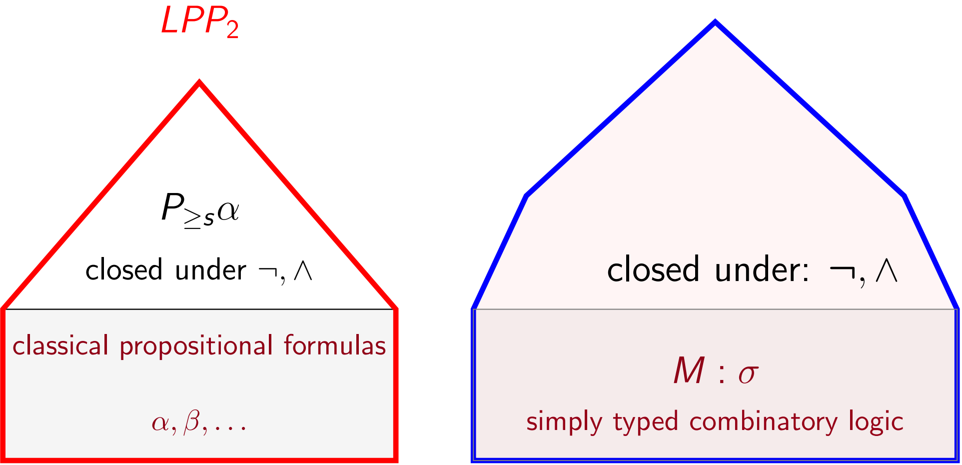

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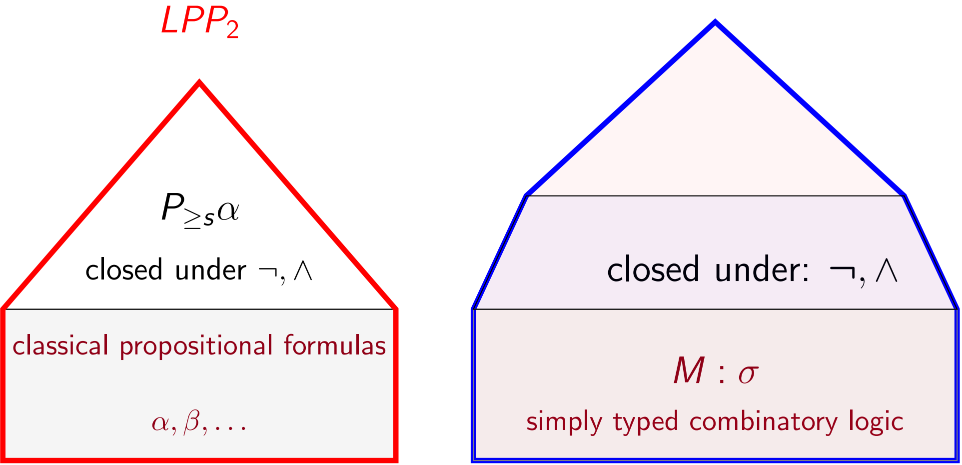

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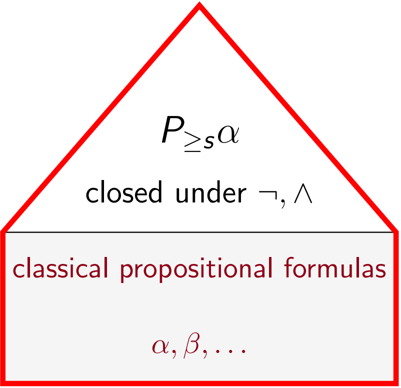

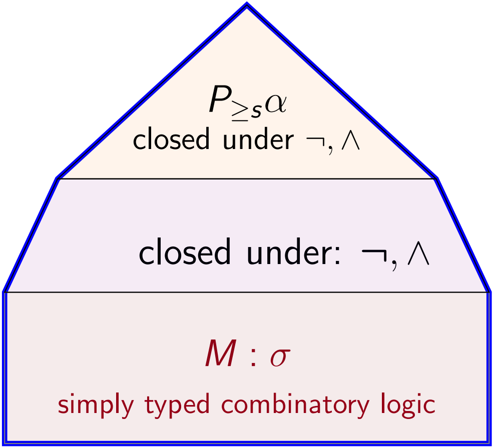

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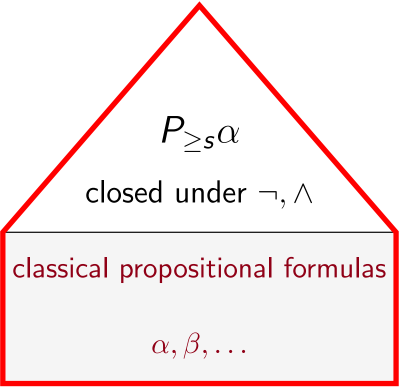

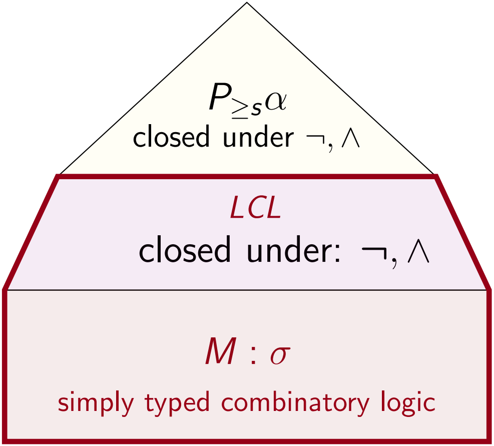

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