Various notions of computability of subsets of topological and metric spaces (2)

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23/09/2021

Let (X, d, α) be a computable metric space, $A, B \subseteq X$ and $\varepsilon > 0$. We write $A \preceq_\varepsilon B$ if for all $a \in A$, there exists $b \in B$ such that $d(a, b) < \varepsilon$. We write $A \approx_\varepsilon B$ if $A \preceq_\varepsilon B$ and $B \preceq_\varepsilon A$.

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$N \subseteq X$ is *computable* if N is compact and there is a recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$, $N \approx_{2^{-k}} \Lambda_{\varphi(k)}$.

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When (X, d) is complete, we get that N is compact at a discount:

Lemma

Let (X, d, α) be a computable metric space such that (X, d) is complete. If N is closed in (X, d) and there is a recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$, $N \approx_{2^{-k}} \Lambda_{\varphi(k)}$, then N is computable in (X, d, α).

Proof.

Since N is a closed subset of a complete metric space, N is also complete. Additionally, for all $k \in \mathbb{N}$, there exists a finite set $\Lambda_{\varphi(k)}$ such that $N \preceq_{2^{-k}} \Lambda_{\varphi(k)}$, so N is totally bounded. These two properties of N imply that N is compact. □

We say that $M \subseteq X$ is *computable from above* in (X, d, α) if there is a recursive $f : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$, $M \preceq_{2^{-k}} \Lambda_{f(k)}$.

$N \subseteq X$ is *computable* if N is compact and there is a recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$, $N \approx_{2^{-k}} \Lambda_{\varphi(k)}$.

This is the theorem we want to prove next:

Theorem

Let (X, d, α) be a computable metric space such that (X, d) is complete. Then $M \subseteq X$ is computable from above in (X, d, α) if and only if there exists a computable $N \supseteq M$.

Theorem

Let (X, d, α) be a computable metric space such that (X, d) is complete. Then $M \subseteq X$ is computable from above in (X, d, α) if and only if there exists a computable $N \supseteq M$.

Proof.

$\boxed{\Leftarrow}$ Trivial. For all k, $N \approx_{2^{-k}} \Lambda_{f(k)}$ implies $N \preceq_{2^{-k}} \Lambda_{f(k)}$, which in turn implies $M \preceq_{2^{-k}} \Lambda_{f(k)}$ since $M \subseteq N$.

$\boxed{\Rightarrow}$ We define the (c.e.) relation $\Gamma \subseteq \mathbb{N}^3$ with $\Gamma(u, k, v) :\Longleftrightarrow$

$$(\forall j \in [u])(j \in [v]) \wedge \tag{Γ_1}$$
$$(\forall j \in [v])(\exists j' \in [u])\big(d(\alpha_j, \alpha_{j'}) < 3 \cdot 2^{-k}\big) \wedge \tag{Γ_2}$$
$$(\forall j \in [f(k+1)])(\forall j' \in [u])\big(d(\alpha_j, \alpha_{j'}) \leq \tfrac{3}{2} \cdot 2^{-k} \to j \in [v]\big). \tag{Γ_3}$$

$\Lambda_u \subseteq \Lambda_v$	$\Lambda_v \preceq_{3 \cdot 2^{-k}} \Lambda_u$	given by f in next step and
no one left behind	goes in $\Rightarrow$ close	really close $\Rightarrow$ goes in

$$(\forall j \in [u])(j \in [v]) \wedge \tag{Γ_1}$$

$$(\forall j \in [v])(\exists j' \in [u])\big(d(\alpha_j, \alpha_{j'}) < 3 \cdot 2^{-k}\big) \wedge \tag{Γ_2}$$

$$(\forall j \in [f(k+1)])(\forall j' \in [u])\big(d(\alpha_j, \alpha_{j'}) \leq \tfrac{3}{2} \cdot 2^{-k} \to j \in [v]\big). \tag{Γ_3}$$

We can prove that $\forall u \, \forall k \, \exists v \, \Gamma(u, k, v)$: simply take v such that

$$[v] = [u] \cup \big\{ j \in [f(k+1)] : d(\alpha_j, \Lambda_u) < 2 \cdot 2^{-k} \big\}.$$

Hence there exists a recursive function φ such that for all u and k we have $\Gamma\big(u, k, \varphi(u, k)\big)$. We define by primitive recursion:

$$\psi(0) := f(0),$$
$$\psi(k+1) := \varphi\big(\psi(k), k\big).$$

The function ψ is recursive and for all k, $\Gamma\big(\psi(k), k, \psi(k+1)\big)$ holds. Let

$$N := \overline{\bigcup_{k \in \mathbb{N}} \Lambda_{\psi(k)}}.$$

$$(\forall j \in [u])(j \in [v]) \tag{Γ_1}$$

$$(\forall j \in [v])(\exists j' \in [u])\big(d(\alpha_j, \alpha_{j'}) < 3 \cdot 2^{-k}\big) \tag{Γ_2}$$

$$\Lambda_{\psi(k)} \approx_{3 \cdot 2^{-k}} \Lambda_{\psi(k+1)} \text{ (due to } (\Gamma_1),(\Gamma_2))$$

$$\Lambda_{\psi(k)} \approx_{3 \cdot 2^{-k}} \Lambda_{\psi(k+1)} \approx_{3 \cdot 2^{-k-1}} \Lambda_{\psi(k+2)} \approx_{3 \cdot 2^{-k-2}} \Lambda_{\psi(k+3)} \cdots$$

$$\Lambda_{\psi(k_1)} \approx_{3 \cdot 2^{-k_1}(1+\frac{1}{2}+\frac{1}{4}+\ldots)} \Lambda_{\psi(k_2)} \text{ when } k_1 < k_2$$

$$\Lambda_{\psi(k_1)} \approx_{3 \cdot 2^{1-k_1}} \Lambda_{\psi(k_2)} \text{ when } k_1 < k_2$$

$$\Lambda_{\psi(k_1)} \approx_{3 \cdot 2^{1-k_1}} \bigcup_k \Lambda_{\psi(k)} \text{ (use } (\Gamma_1) \text{ for } k \leq k_1)$$

$$\Lambda_{\psi(k_1)} \approx_{4 \cdot 2^{1-k_1}} \overline{\bigcup_k \Lambda_{\psi(k)}}$$

$$\Lambda_{\psi(k_1)} \approx_{2^{-(k_1-3)}} N$$

Hence, for all $k \in \mathbb{N}$ we have

$$\Lambda_{\psi(k+3)} \approx_{2^{-k}} N.$$

Since $k \mapsto \psi(k+3)$ is recursive, by our Lemma N is computable.

$$(\forall j \in [f(k+1)])(\forall j' \in [u])\big(d(\alpha_j, \alpha_{j'}) \leq \tfrac{3}{2} \cdot 2^{-k} \to j \in [v]\big) \quad (\Gamma_3)$$

Finally, to prove $M \subseteq N$, we first prove that for all k, $M \preceq_{2^{-k}} \Lambda_{\psi(k)}$ holds. We do so by induction on k:

$\boxed{k = 0}$ We have that $M \preceq_{2^{-k}} \Lambda_{f(0)} = \Lambda_{\psi(0)}$.

$\boxed{k+1}$ Let $x \in M$. From $M \preceq_{2^{-(k+1)}} \Lambda_{f(k+1)}$, there exists $j \in [f(k+1)]$ such that $d(x, \alpha_j) < 2^{-k-1}$. The induction hypothesis implies there exists $j' \in [\psi(k)]$ such that $d(x, \alpha_{j'}) < 2^{-k}$ holds. We now have

$$d(\alpha_j, \alpha_{j'}) < \tfrac{3}{2} \cdot 2^{-k},$$

so $j \in [\psi(k+1)]$ due to (Γ_3).

To see that $M \subseteq N$, let $x \in M$. For all $k \in \mathbb{N}$, there is an $x_k \in \Lambda_{\psi(k)}$ such that $d(x, x_k) < 2^{-k}$. The sequence (x_k) in $\bigcup_k \Lambda_{\psi(k)}$ converges to x, so $x \in \overline{\bigcup_k \Lambda_{\psi(k)}} = N$. □

Theorem

Let (X, d, α) be a computable metric space. If $K \subseteq X$ is semicomputable, then K is computable from above.

Proof.

We define

$$\Gamma := \{(k, j) \in \mathbb{N}^2 \colon K \subseteq J_j, \rho_i < 2^{-k} \text{ for all } i \in [j]\}.$$

The set K is compact, so it is totally bounded. Hence, for all $k \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $\Gamma(k, j)$ holds. By the selection theorem, there exists a recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$, $\Gamma(k, \varphi(k))$ holds. Hence we have

$$K \preceq_{2^{-k}} \{\lambda_i \colon i \in [\varphi(k)]\}.$$

Let τ be the bijective enumeration of $\mathbb{N}^2$ and τ_1 its first coordinate map. Then by the above we get

$$K \preceq_{2^{-k}} \{\alpha_i \colon i \in \tau_1([\varphi(k)])\}.$$

The function $k \mapsto \tau_1([\varphi(k)])$ is c.f.v., so there exists a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$, $[f(k)] = \tau_1([\varphi(k)])$. Therefore,

$$K \preceq_{2^{-k}} \{\alpha_i \colon i \in [f(k)]\} = \Lambda_{f(k)}. \qquad \square$$

Note that the converse does not hold in general by a simple cardinality argument. There exist only countably many semicomputable sets (as there are only countably many recursive functions), but e.g. every one-element subset of $[0, 1]$ is computable from above (due to the trivial direction of the first theorem).