Mathematical Institute
of the Serbian Academy of Sciences and Arts

Minimal models for graphs-related operadic algebras

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Operads in proof theory

$$A \vdash B \qquad \frac{}{A \vdash A}\,\texttt{ax} \qquad \frac{A \vdash B \quad B \vdash C}{A \vdash C}\,\texttt{cut}$$

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a category

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$$A_1, \ldots, A_n \vdash B \qquad \frac{}{A \vdash A}\,\texttt{ax} \qquad \frac{\Gamma \vdash A \qquad \Delta_1, A, \Delta_2 \vdash B}{\Delta_1, \Gamma, \Delta_2 \vdash B}\,\texttt{cut}$$

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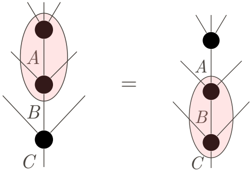

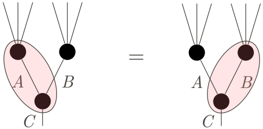

a multicategory

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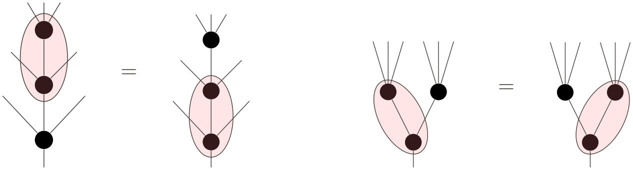

an operad

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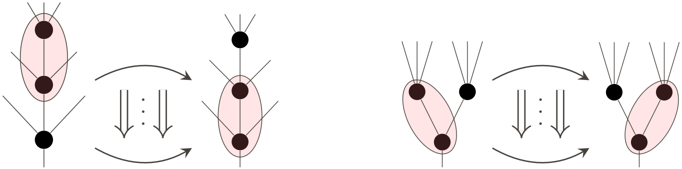

a strongly homotopy operad

Goal & Methods used

Strongly homotopy modular operads within the framework of operadic categories.

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$$H_n(\mathfrak{M}_{\mathtt{Grc}}(\Gamma)) = H_n(\mathbb{1}_{\mathtt{Grc}}(\Gamma)) = \begin{cases} \mathbb{k}, & n = 0 \\ 0, & n \geq 1 \end{cases}$$

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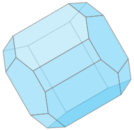
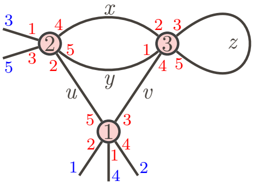


Modular operads: the terminal operad $1_{\mathtt{Grc}}$



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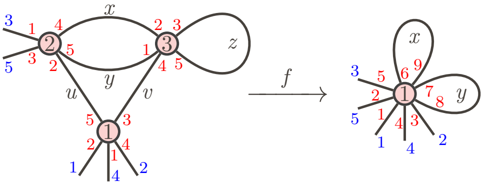

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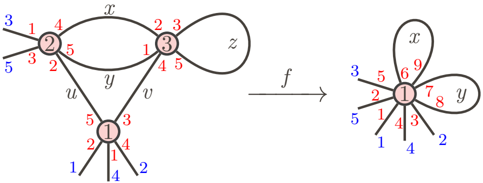
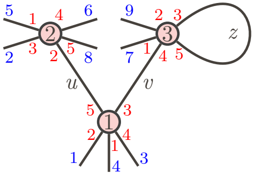


$$f^{-1}(1) =$$



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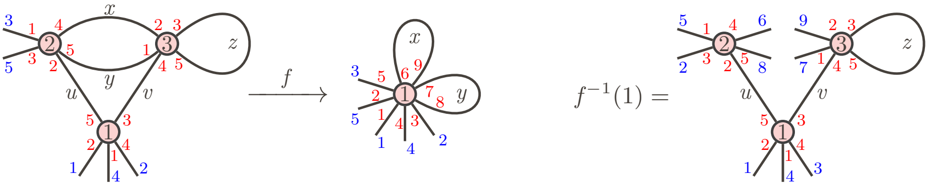

$$1_{\mathtt{Grc}}(\Gamma) = \Bbbk, \quad \forall\, \Gamma \in \mathtt{Grc}$$

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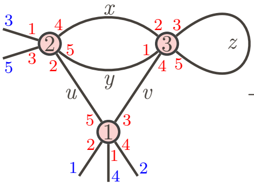
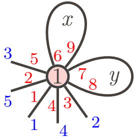
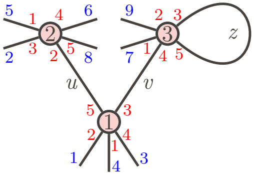


$\xrightarrow{\ f\ }$



$f^{-1}(1) =$ 

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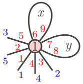
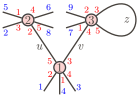
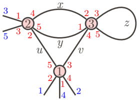
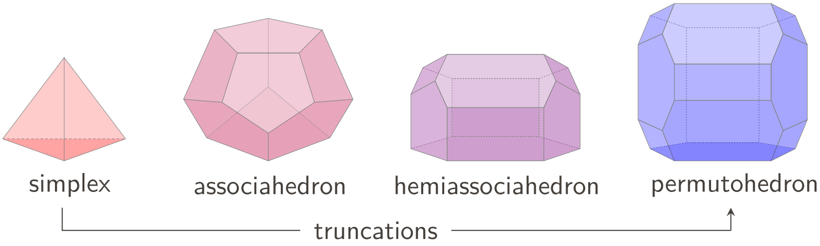
$\circ_f \; : \; 1_{\mathtt{Grc}}\Bigg($  $\Bigg) \otimes 1_{\mathtt{Grc}}\Bigg($  $\Bigg) \to 1_{\mathtt{Grc}}\Bigg($  $\Bigg)$

$$\circ_f : \mathbb{k} \otimes \mathbb{k} \xrightarrow{\cong} \mathbb{k}$$

Hypergraph polytopes, a.k.a. nestohedra

K. Došen, Z. Petrić
Hypergraph polytopes
Topology and its Applications 158, pp. 1405–1444, 2011

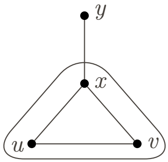
P.-L. Curien, J. Obradović, J. Ivanović
Syntactic aspects of hypergraph polytopes
Journal of Homotopy and Related Structures 14, pp. 235–279, 2019



Hypergraph terminology

$$H = \{x, y, u, v\}$$

$$\mathbf{H} = \{\{x\}, \{y\}, \{u\}, \{v\}, \{x, u\}, \{u, v\}, \{x, v\}, \{x, y\}, \{x, u, v\}\}$$



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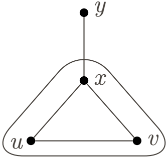

$\mathbf{H}$ is **saturated**: $(\forall X, Y \in \mathbf{H})\ X \cap Y \neq \emptyset \Rightarrow X \cup Y \in \mathbf{H}$

$$Sat(\mathbf{H}) = \mathbf{H} \cup \{\{x, y, u\}, \{x, y, v\}, \{x, y, z, u\}\}$$

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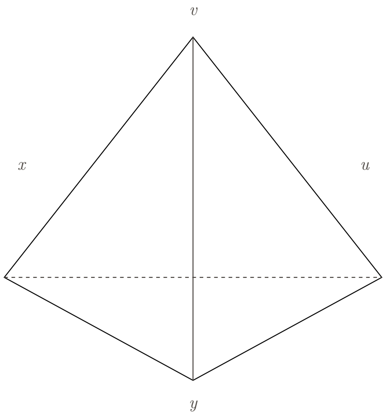

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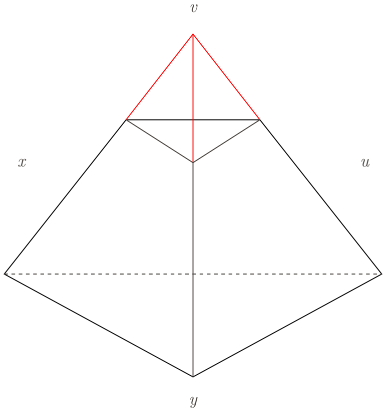

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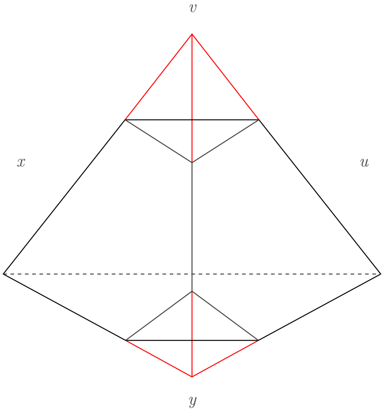

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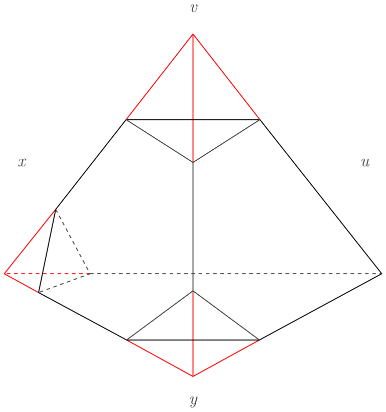

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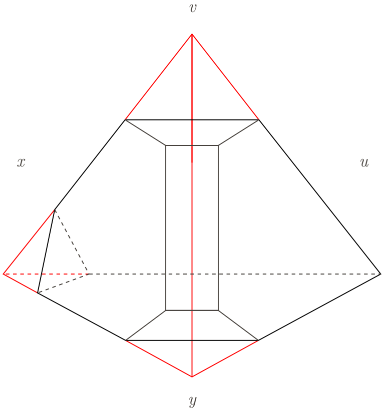

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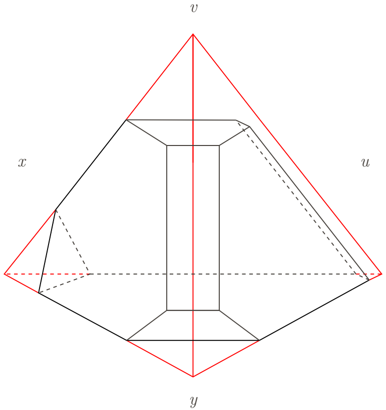

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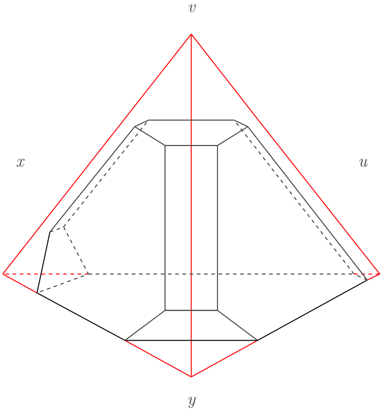

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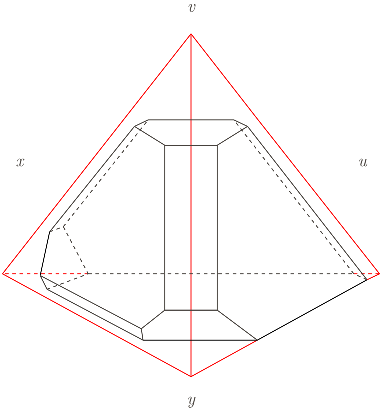

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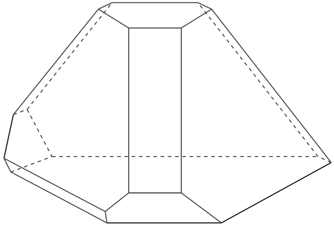
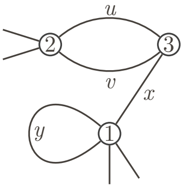
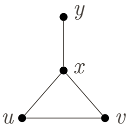


Strongly homotopy modular operads: the minimal model $\mathfrak{M}_{\mathtt{Grc}}$ of $1_{\mathtt{Grc}}$

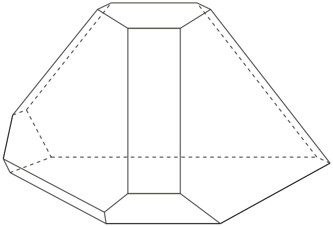
$\Gamma =$





$= \mathbf{H}_\Gamma$

$\mathfrak{M}_{\mathtt{Grc}}(\Gamma) \quad =$

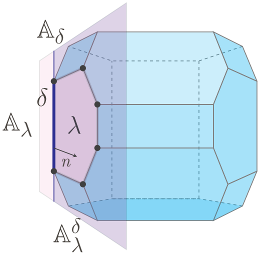


The ingenious lemma

The faces of $\mathcal{G}(\mathbf{H}_\Gamma)$ can be oriented, so that $\mathfrak{M}_{\mathtt{Grc}}(\Gamma)$ is the cellular chain complex $(C_*(\mathcal{G}(\mathbf{H}_\Gamma)), \partial_*)$ of free abelian groups $C_k(\mathcal{G}(\mathbf{H}_\Gamma))$ generated by k-dimensional faces of $\mathcal{G}(\mathbf{H}_\Gamma)$, whose differential ∂_* is given by

$$\partial_*(\lambda) := \sum_{\delta \lessdot \lambda} \eta_\lambda^\delta \cdot \delta,$$

where $\eta_\lambda^\delta := +1$ if δ is oriented compatibly with λ and $\eta_\lambda^\delta := -1$ otherwise.



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Proof. Pick an orientation of the n-dimensional face. Pick a $(k-1)$-dimensional face a and choose $a \lessdot e$. If a occurs in $\partial(e)$ with the $+$, give it the compatible orientation; otherwise, give it the orientation opposite to the compatible one.

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This receipt does not depend on the choice of e, thanks to:

💎 If a is a $(k-1)$-dimensional face of $\mathcal{G}(\mathbf{H})$ such that $a \lessdot e', e''$, then there exists a $(k+1)$-dimensional face h such that $e', e'' \lessdot h$.

Thank you!

This work has been supported by the Praemium Academiae of M. Markl and RVO:67985840.