Chapter 1

Integrali funkcija više promenljivih

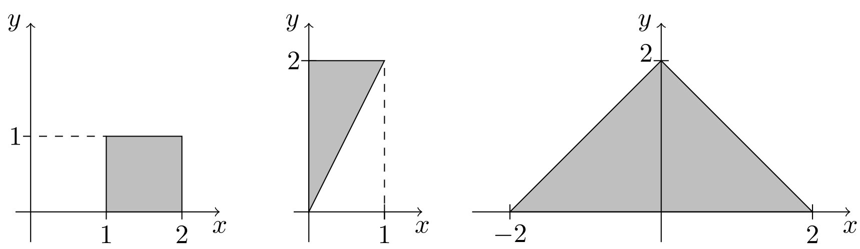
1.1 Dvostruki integrali

1. **Odrediti granice integracije dvostrukog integrala $\iint\limits_D f(x,y)dxdy$,**

   ako je oblast D ograničena

   (a) **pravama $x=1, x=2, y=1$ i x-osom,**
   (b) **pravama $x=0, y=2$ i $y=2x$,**
   (c) **pravama $y=2-x, y=2+x$ i x-osom,**

   kao što je predstavljeno na sledećoj slici:



(a) (b) (c)

Rešenje:

(a) Oblast integracije je

$$D=\{(x,y)\in\mathbb{R}^2 : 1\le x\le 2, 0\le y\le 1\},$$

odakle je

$$\iint_D f(x,y)dxdy = \int_0^1 \left(\int_1^2 f(x,y)dx \right) dy = \int_1^2 \left(\int_0^1 f(x,y)dy \right) dx.$$

(b) Oblast integracije je

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, 2x \le y \le 2\},$$

odakle je

$$\iint_D f(x,y)dxdy = \int_0^1 \left(\int_{2x}^2 f(x,y)dy \right) dx.$$

Primetimo da u ovom slučaju moramo odrediti nove granice integracije, da bismo zamenili redosled integracije. Ista oblast može se drugačije opisati kao

$$D = \left\{(x,y) \in \mathbb{R}^2 : 0 \le x \le \frac{y}{2}, 0 \le y \le 2\right\},$$

te je

$$\iint_D f(x,y)dxdy = \int_0^2 \left(\int_0^{\frac{y}{2}} f(x,y)dx \right) dy.$$

(c) Slično razmatranju u prethodnom primeru, ako je oblast D opisana sa

$$D = \{(x,y) \in \mathbb{R}^2 : y-2 \le x \le 2-y, 0 \le y \le 2\},$$

onda redosled integracije može biti

$$\iint_D f(x,y)dxdy = \int_0^2 \left(\int_{y-2}^{2-y} f(x,y)dx \right) dy.$$

Primetimo da se oblast može opisati i na drugi način. Ako želimo da promenimo redosled integracije, oblast se mora podeliti na dva dela tj. predstaviti kao unija dve podoblasti $D = D_1 \cup D_2$, gde je

$$\begin{aligned} D_1 &= \{(x,y) \in \mathbb{R}^2 : -2 \le x \le 0, 0 \le y \le 2+x\} \\ D_2 &= \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2, 0 \le y \le 2-x\}. \end{aligned}$$

Sada je redosled integracije

$$\begin{aligned} \iint_D f(x,y)dxdy &= \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy \\ &= \int_{-2}^0 \left(\int_0^{2+x} f(x,y)dy \right) dx + \int_0^2 \left(\int_0^{2-x} f(x,y)dy \right) dx. \end{aligned}$$

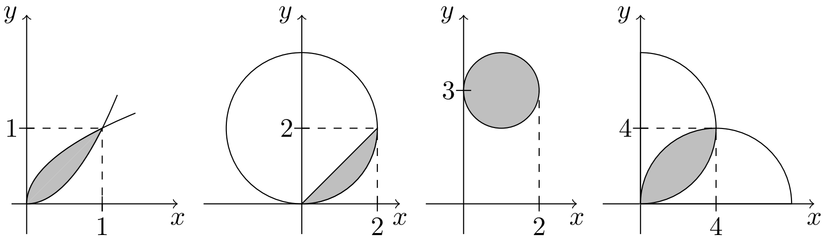
2. **Odrediti granice integracije integrala $\iint\limits_D f(x,y)dxdy$ za dva moguća redosleda integracije ako je:**

   (a) $D = \{(x,y) \in \mathbb{R}^2 : y \leq \sqrt{x}, x \leq \sqrt{y}\}$,

   (b) $D = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-2)^2 \leq 4, y \leq x\}$,

   (c) $D = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y-3)^2 \leq 1\}$,

   (d) $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 8y, x^2 + y^2 \leq 8x\}$.



(a) (b) (c) (d)

Rešenje:

(a) Ako primetimo da se skup D može opisati na sledeća dva načina:

$$\begin{aligned} D &= \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, x^2 \leq y \leq \sqrt{x}\}, \\ D &= \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 1, y^2 \leq x \leq \sqrt{y}\}. \end{aligned}$$

onda zaključujemo da važi:

$$I = \int\limits_0^1 \left(\int\limits_{x^2}^{\sqrt{x}} f(x,y)dy \right) dx = \int\limits_0^1 \left(\int\limits_{y^2}^{\sqrt{y}} f(x,y)dx \right) dy.$$

(b) Tačke preseka prave $y = x$ i kružnice $x^2 + (y-2)^2 = 4$ dobijamo rešavanjem sledećeg sistema jednačina:

$$\begin{aligned} y = x \ \wedge \ x^2 + (y-2)^2 = 4 &\Leftrightarrow y = x \wedge x^2 + (x-2)^2 = 4 \\ &\Leftrightarrow y = x \wedge 2x^2 - 4x = 0 \\ &\Leftrightarrow y = x \wedge x(x-2) = 0 \\ &\Leftrightarrow x_1 = y_1 = 0, x_2 = y_2 = 2. \end{aligned}$$

Jednačina dela kružnice $x^2 + (y-2)^2 = 4$ za koji je $0 \leq y \leq 2$ je oblika

$$y - 2 = -\sqrt{4 - x^2} \Leftrightarrow y = 2 - \sqrt{4 - x^2}.$$

Slično, jednačina dela kružnice $x^2 + (y-2)^2 = 4$ za koji je $0 \le x \le 2$ je oblika

$$x = +\sqrt{4-(y-2)^2} \Leftrightarrow x = \sqrt{4y-y^2}.$$

Odatle je

$$I = \int_0^2 \left(\int_{2-\sqrt{4-x^2}}^{x} f(x,y)dy \right) dx = \int_0^2 \left(\int_{y}^{\sqrt{4y-y^2}} f(x,y)dx \right) dy.$$

(c) $I = \int_0^2 \left(\int_{3-\sqrt{2x-x^2}}^{3+\sqrt{2x-x^2}} f(x,y)dy \right) dx = \int_2^4 \left(\int_{1-\sqrt{1-(y-3)^2}}^{1+\sqrt{1-(y-3)^2}} f(x,y)dx \right) dy.$

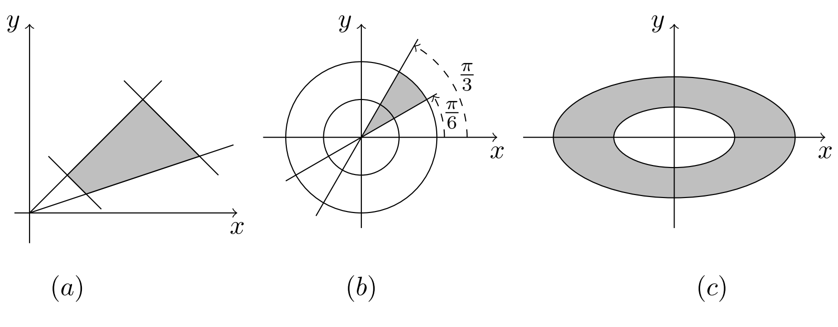
(d) $I = \int_0^4 \left(\int_{4-\sqrt{16-x^2}}^{\sqrt{8x-x^2}} f(x,y)dy \right) dx = \int_0^4 \left(\int_{4-\sqrt{16-y^2}}^{\sqrt{8y-y^2}} f(x,y)dx \right) dy.$

3. **Uvesti pogodnu transformaciju koordinata i izračunati njen Jakobijan, ako je**

(a) $\boldsymbol{D = \left\{(x,y) \in \mathbb{R}^2 : y \le x \le 3y, 1 \le x+y \le 3\right\}}$**,**

(b) $\boldsymbol{D = \left\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 9, \frac{x}{\sqrt{3}} \le y \le \sqrt{3}x\right\}}$**,**

(c) $\boldsymbol{D = \left\{(x,y) \in \mathbb{R}^2 : 1 \le \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 4\right\}}$**,** $\boldsymbol{a, b \in \mathbb{R}}$**.**



Rešenje:

(a) Ako uvedemo smenu $u = \dfrac{x}{y}$ i $v = x+y$, onda je $u \in [1,3]$ i $v \in [1,3]$. Iz smene dobijamo da je $x = \dfrac{uv}{u+1}$ i $y = \dfrac{v}{u+1}$, odakle je Jakobijan transformacije

$$\begin{aligned} J(u,v) &= \begin{vmatrix} \frac{v}{(u+1)^2} & \frac{u}{u+1} \\ -\frac{v}{(u+1)^2} & \frac{1}{u+1} \end{vmatrix} \\ &= \frac{v}{(u+1)^2}\frac{1}{u+1} - \frac{u}{u+1}\left(-\frac{v}{(u+1)^2}\right) = \frac{v}{(u+1)^2}. \end{aligned}$$

(b) Sa slike vidimo da je oblast D deo kruga koji leži u prvom kvadrantu između pravih $y = \frac{1}{\sqrt{3}}x$ i $y = \sqrt{3}x$. Uvešćemo transformaciju Dekartovih u polarne koordinate $x = \rho\cos\varphi, y = \rho\sin\varphi$, pri čemu $\rho \in [0,3]$ i $\varphi \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$.
Jakobijan transformacije je

$$J(\rho,\varphi) = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho\cos^2\varphi + \rho\sin^2\varphi = \rho.$$

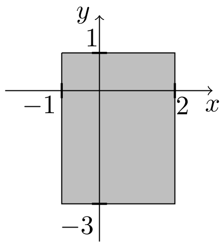
(c) Data oblast se nalazi u prstenu između dve elipse, kao što je to ilustrovano na slici. U ovom slučaju možemo uvesti smenu $\frac{x}{a} = \rho\cos\varphi, \frac{y}{b} = \rho\sin\varphi$, pri čemu $\rho \in [1,2],\ \varphi \in [0,2\pi]$ i $a,b \in \mathbb{R}$.
Jakobijan transformacije je

$$J(\rho,\varphi) = \begin{vmatrix} a\cos\varphi & -a\rho\sin\varphi \\ b\sin\varphi & b\rho\cos\varphi \end{vmatrix} = ab\rho\cos^2\varphi + ab\rho\sin^2\varphi = ab\rho.$$

4. **Izračunati $\iint\limits_D 2x^3y + xydxdy$ ako je $D = \left\{(x,y) \in \mathbb{R}^2 : -1 \le x \le 2, -3 \le y \le 1\right\}$.**

Rešenje:

$$\begin{aligned} I &= \int_{-3}^{1}\left(\int_{-1}^{2}(2x^3y + xy)dx\right)dy \\ &= \int_{-3}^{1}\left(\frac{x^4y}{2} + \frac{x^2y}{2}\right)\Bigg|_{x=-1}^{x=2} dy \\ &= \int_{-3}^{1} 9ydy = -36. \end{aligned}$$



5. **Izračunati $I = \iint\limits_D \rho^2\varphi d\rho d\varphi$ ako je $D = \left\{(\rho,\varphi) \in [0,2] \times [\frac{\pi}{6}, \frac{\pi}{3}]\right\}$.**

Rešenje: Funkcija $f(\rho,\varphi) = \rho^2\varphi$ je proizvod funkcija jedne promenljive koje zavise od ρ i φ. Osim toga, oblast D je pravougaona, zbog čega se dati integral može izračunati kao proizvod dva određena integrala.

$$I = \iint\limits_D f(\rho,\varphi)d\rho d\varphi = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\varphi d\varphi \cdot \int_0^2 \rho^2 d\rho = \frac{\varphi^2}{2}\Bigg|_{\varphi=\frac{\pi}{6}}^{\varphi=\frac{\pi}{3}} \cdot \frac{\rho^3}{3}\Bigg|_{\rho=0}^{\rho=2} = \frac{\pi^2}{9}.$$

6. **Izračunati dvostruki integral $I = \iint\limits_D x dx dy$ ako je**

   (a) $\boldsymbol{D = \{(x,y) \in \mathbb{R}^2 : 2 \leq x \leq 3, \frac{1}{x} \leq y \leq x^2\}}$,

   (b) $\boldsymbol{D = \{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 25 - y^2, -3 \leq y \leq 0\}}$,

   (c) $\boldsymbol{D = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y-3)^2 \leq 1\}}$.

   Rešenje:

$$\text{(a)}\quad I = \int_2^3 \left(\int_{\frac{1}{x}}^{x^2} x dy \right) dx = \int_2^3 \left(x^3 - 1\right) dx = \left(\frac{x^4}{4} - x \right)\Bigg|_{x=2}^{x=3} = \frac{61}{4}$$

$$\begin{aligned}\text{(b)}\quad I &= \int_{-3}^{0} \left(\int_1^{25-y^2} x dx \right) dy = \frac{1}{2} \int_{-3}^{0} \left((25-y^2)^2 - 1\right) dy \\ &= \frac{1}{2} \int_{-3}^{0} \left(624 - 50y^2 + y^4\right) dy = \frac{1}{2} \left(624\, y - \frac{50}{3} y^3 + \frac{y^5}{5} \right)\Bigg|_{y=-3}^{y=0} \\ &= \tfrac{1}{2}\left(1872 - 450 + \tfrac{243}{5}\right) = \tfrac{7353}{10}.\end{aligned}$$

   (c) Integral rešavamo transformacijom koordinata

$$x - 1 = \rho \cos\varphi,\ y - 3 = \rho \sin\varphi, \rho \in [0,1], \varphi \in [0, 2\pi].$$

   Jakobijan transformacije je $J = \rho$. Time se integral svodi na:

$$\begin{aligned} I &= \int_0^{2\pi} \left(\int_0^1 (1 + \rho\cos\varphi)\rho d\rho \right) d\varphi = \int_0^{2\pi} \left(\frac{\rho^2}{2} + \frac{\rho^3}{3} \cos\varphi \right)\Bigg|_{\rho=0}^{\rho=1} d\varphi \\ &= \frac{1}{6} \int_0^{2\pi} (3 + 2\cos\varphi) d\varphi = \frac{1}{6}(3 \cdot 2\pi) = \pi. \end{aligned}$$

7. **Izračunati površinu figure:**

   (a) $\boldsymbol{D = \left\{(x,y) \in \mathbb{R}^2 : 9 \leq x^2 + y^2 \leq 16\right\}}$,

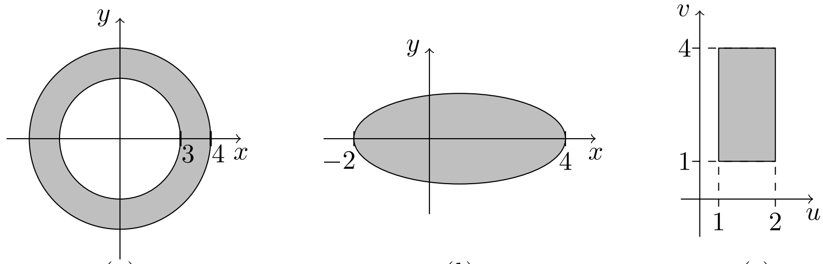
   (b) $\boldsymbol{D = \left\{(x,y) \in \mathbb{R}^2 : 4(x-1)^2 + 9y^2 \leq 36\right\}}$ **i**

   (c) $\boldsymbol{D = \left\{(x,y) \in \mathbb{R}^2 : \frac{x}{2} \leq y \leq 2x, 1 \leq xy \leq 4, x \geq 0, y \geq 0\right\}}$.

   Rešenje:

   (a) Oblast čiju površinu treba izračunati je kružni prsten. Uvodimo transformaciju

$$x = \rho\cos\varphi,\ y = \rho\sin\varphi, \quad \rho \in [3,4],\ \varphi \in [0, 2\pi].$$



(a) (b) (c)

Jakobijan transformacije je $J=\rho$. Površina date oblasti je tada:

$$\Delta D=\iint\limits_D dxdy=\int\limits_0^{2\pi}d\varphi\cdot\int\limits_3^4\rho d\rho=2\pi\cdot\left.\frac{\rho^2}{2}\right|_3^4=7\pi.$$

(b) Kako je

$$4(x-1)^2+9y^2\le 36\Leftrightarrow\frac{(x-1)^2}{9}+\frac{y^2}{4}\le 1,$$

integral možemo izračunati uvođenjem transformacije

$$\frac{x-1}{3}=\rho\cos\varphi,\quad \frac{y}{2}=\rho\sin\varphi,\quad \rho\in[0,1],\quad \varphi\in[0,2\pi].$$

Odatle zaključujemo da je

$$x=1+3\,\rho\cos\varphi,\quad y=2\,\rho\sin\varphi,$$

i Jakobijan transformacije je $J=6\rho$. Površina date oblasti je

$$\Delta D=\iint\limits_D dxdy=6\int\limits_0^{2\pi}d\varphi\cdot\int\limits_0^1\rho d\rho=6\cdot 2\pi\cdot\left.\frac{\rho^2}{2}\right|_0^1=6\pi.$$

(c) Ako uvedemo transformaciju koordinata $u=\frac{y}{x}, v=xy$ tada oblast integracije postaje pravougaona oblast u uv-ravni, za koju $u\in\left[\frac{1}{2},2\right]$ i $v\in[1,4]$. Iz transformacije dobijamo da je $x=\sqrt{\frac{v}{u}}$ i $y=u\sqrt{\frac{v}{u}}$, odakle je Jakobijan transformacije

$$J(u,v)=\begin{vmatrix}-\frac{1}{2u}\sqrt{\frac{v}{u}} & \frac{1}{2v}\sqrt{\frac{v}{u}}\\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}}\end{vmatrix}=-\frac{1}{4u}-\frac{1}{4v}\cdot\frac{v}{u}=-\frac{1}{2u}.$$

Tada je tražena površina:

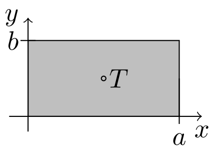
$$\Delta D=\iint\limits_D dx\,dy=\int\limits_{\frac{1}{2}}^{2}\left(\int\limits_1^4\left|-\frac{1}{2u}\right|dv\right)du=\frac{3}{2}\left.(\ln u)\right|_{\frac{1}{2}}^{2}=3\ln 2.$$

8. **Izračunati masu i težište homogene ploče**

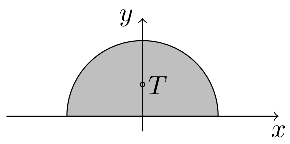
   (a) $\boldsymbol{D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq a, 0 \leq y \leq b\}}$ **i**

   (b) $\boldsymbol{D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9, y \geq 0\}}$**.**

   gustine $\boldsymbol{\mu(x,y) = 1,\ (x,y) \in D}$**.**



(a)



(b)

Rešenje:

(a) Masa date ploče je

$$m = \iint\limits_D \mu\, dxdy = \mu \cdot \Delta D = \mu ab.$$

Težište ploče je tačka $T(x_T, y_T)$, čije koordinate su

$$x_T = \frac{1}{m} \iint\limits_D \mu\, x\, dxdy = \frac{1}{ab} \iint\limits_D xdxdy = \frac{1}{ab} \int\limits_0^b \left(\int\limits_0^a xdx \right) dy$$

$$= \frac{1}{ab} \int\limits_0^b \frac{x^2}{2} \Bigg|_{y=0}^{y=a} dy = \frac{1}{ab} \frac{a^2}{2} \int\limits_0^b dy = \frac{1}{ab} \frac{a^2 b}{2} = \frac{a}{2},$$

$$y_T = \frac{1}{m} \iint\limits_D \mu\, y\, dxdy = \frac{1}{ab} \iint\limits_D ydxdy = \frac{1}{ab} \int\limits_0^a \left(\int\limits_0^b ydy \right) dx$$

$$= \frac{1}{ab} \int\limits_0^a \frac{y^2}{2} \Bigg|_{x=0}^{x=b} dx = \frac{1}{ab} \frac{b^2}{2} \int\limits_0^a dx = \frac{1}{ab} \frac{b^2 a}{2} = \frac{b}{2}.$$

(b) Zbog specifičnosti oblasti prelazimo na polarne koordinate:

$$x = \rho \cos\varphi,\ y = \rho \sin\varphi,\ \rho \in [0,3],\ \varphi \in [0,\pi].$$

Odatle< je masa

$$m = \iint\limits_D dxdy = \int\limits_0^\pi d\varphi \cdot \int\limits_0^3 \rho d\rho = \varphi \Big|_0^\pi \cdot \frac{\rho^2}{2} \Big|_0^3 = \frac{9\pi}{2}.$$

$$x_t = \frac{1}{m}\iint\limits_D x dx dy = \frac{2}{9\pi}\int\limits_0^{\pi}\left(\int\limits_0^3 \rho^2\cos\varphi d\rho\right)d\varphi$$
$$= \frac{2}{9\pi}\int\limits_0^{\pi}\frac{\rho^3}{3}\Bigg|_{\rho=0}^{\rho=3}\cos\varphi d\varphi = \frac{2}{9\pi}9\int\limits_0^{\pi}\cos\varphi d\varphi = \frac{2}{\pi}\sin\varphi\Big|_0^{\pi} = 0,$$

$$y_t = \frac{1}{m}\iint\limits_D y dx dy = \frac{2}{9\pi}\int\limits_0^{\pi}\left(\int\limits_0^3 \rho^2\sin\varphi d\rho\right)d\varphi$$
$$= \frac{2}{9\pi}\int\limits_0^{\pi}\frac{\rho^3}{3}\Bigg|_0^3\sin\varphi d\varphi = \frac{2}{9\pi}\cdot 9\int\limits_0^{\pi}\sin\varphi d\varphi = \frac{2}{\pi}\left(-\cos\varphi\right)\Big|_0^{\pi} = \frac{4}{\pi}.$$

9. **Izračunati masu i težište ploče**

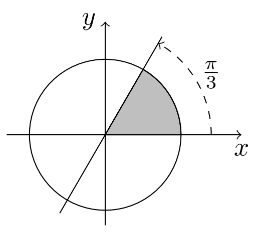
$$\boldsymbol{D = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le 25, 0 \le y \le \sqrt{3}x\}}$$

gustine $\mu(x,y) = x^2+y^2$, $(x,y)\in D$.

Rešenje:

Oblast integracije prikazana je na slici. Uvodimo smenu:

$$x = \rho\cos\varphi,\ y = \rho\sin\varphi,$$
$$\rho\in[0,5],\ \varphi\in\left[0,\tfrac{\pi}{3}\right].$$



Tada je:

$$m = \iint\limits_D (x^2+y^2)dxdy = \int\limits_0^{\frac{\pi}{3}}\left(\int\limits_0^5(\rho^2\cos^2\varphi+\rho^2\sin^2\varphi)\rho d\rho\right)d\varphi$$
$$= \int\limits_0^{\frac{\pi}{3}}\left(\int\limits_0^5\rho^3 d\rho\right)d\varphi = \int\limits_0^{\frac{\pi}{3}}d\varphi\cdot\int\limits_0^5\rho^3 d\rho = \varphi\Big|_0^{\frac{\pi}{3}}\cdot\frac{1}{4}\rho^4\Big|_0^5 = \frac{625\pi}{12},$$

$$x_T = \frac{1}{m}\iint_D x\mu(x,y)dxdy = \frac{12}{625\pi}\int_0^{\frac{\pi}{3}}\left(\int_0^5 \rho^4\cos\varphi d\rho\right)d\varphi$$
$$= \frac{12}{625\pi}\int_0^{\frac{\pi}{3}}\cos\varphi d\varphi\cdot\int_0^5\rho^4 d\rho = \frac{12}{625\pi}\sin\varphi\Big|_0^{\frac{\pi}{3}}\cdot\frac{\rho^5}{5}\Big|_0^5 = \frac{6\sqrt{3}}{\pi},$$

$$y_T = \frac{1}{m}\iint_D y\mu(x,y)dxdy = \frac{12}{625\pi}\int_0^{\frac{\pi}{3}}\left(\int_0^5 \rho^4\sin\varphi d\rho\right)d\varphi$$
$$= \frac{12}{625\pi}\int_0^{\frac{\pi}{3}}\sin\varphi d\varphi\int_0^5\rho^4 d\rho = \frac{12}{625\pi}\left(-\cos\varphi\right)\big|_0^{\frac{\pi}{3}}\frac{\rho^5}{5}\Big|_0^5 = \frac{6}{\pi}.$$

Dakle, masa ploče je $m = \frac{625\pi}{12}$, dok su koordinate težišta ploče $T\left(\frac{6\sqrt{3}}{\pi}, \frac{6}{\pi}\right)$.

10. **Izračunati zapreminu oblasti:**

 (a) $\boldsymbol{V = \{(x,y,z)\in\mathbb{R}^3 : -2\le x\le 2, -3\le y\le 1, 0\le z\le 5\}}$,

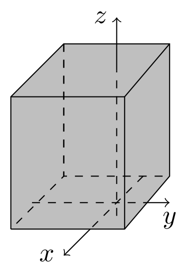
 (b) $\boldsymbol{V = \{(x,y,z)\in\mathbb{R}^3 : z-5\le -\sqrt{x^2+y^2}, z\ge 1\}}$,

 (c) $\boldsymbol{V = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le 16, 0\le z\le 9-x^2-y^2\}}$,

 (d) $\boldsymbol{V = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le 1,}$
 $\boldsymbol{x^2+y^2-2\le z\le 2-x^2-y^2\}}$.

Resenje:

(a) Oblast V prikazana na slici je kvadar u $\mathbb{R}^3$.



Zapreminu ćemo izračunati kao dvostruki integral funkcije $z = 5$ nad projekcijom D na xy-ravan:

$$D = \{(x,y)\in\mathbb{R}^2 : -2\le x\le 2, -3\le y\le 1\}.$$

Znači,

$$\Delta V = \iint_D 5dxdy = \int_{-3}^1\left(\int_{-2}^2 5dx\right)dy = 5\int_{-3}^1 dy\cdot\int_{-2}^2 dx$$
$$= 5\cdot 4\cdot 4 = 80.$$

(b) Primetimo prvo da za svaku tačku (x, y, z) skupa V važi

$$1 \leq z \leq 5 - \sqrt{x^2 + y^2}.$$

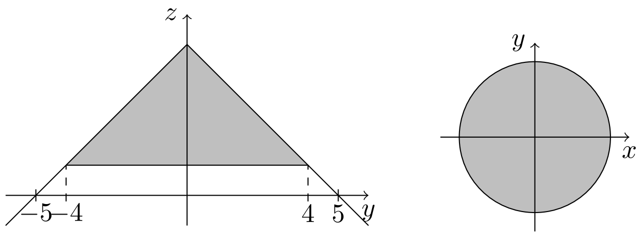
Oblast V čiju zapreminu treba izračunati je ograničena sa gornje strane delom konusa $z = 5 - \sqrt{x^2 + y^2}$, dok je sa donje strane ograničena sa ravni $z = 1$. Dobijeno telo je kupa u $\mathbb{R}^3$, visine 4 i poluprečnika osnove 4. Poluprečnik osnove dobijamo iz preseka ravni i konusa

$$z = 1 \wedge z = 5 - \sqrt{x^2 + y^2} \Leftrightarrow x^2 + y^2 = 4^2.$$

Dakle, projekcija na xy ravan je

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 16\},$$

pa je zapremina dvostuki integral nad D razlike funkcija $z_2(x, y) = 5 - \sqrt{x^2 + y^2}$ i $z_1(x, y) = 1$.



Kako je projekcija centralni krug poluprečnika 4, uvešćemo smenu

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \;\; \rho \in [0, 4], \;\; \varphi \in [0, 2\pi].$$

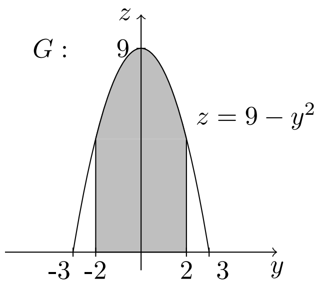
Jakobijan smene je $J = \rho$.

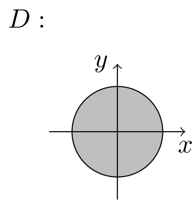
$$\begin{aligned}
\Delta V &= \iint_D (4 - \sqrt{x^2 + y^2}) dx dy = \int_0^{2\pi} \left(\int_0^4 (4 - \rho) \rho d\rho \right) d\varphi \\
&= \int_0^{2\pi} d\varphi \int_0^4 (4\rho - \rho^2) d\rho = 2\pi \left(4\frac{\rho^2}{2} - \frac{\rho^3}{3} \right) \Bigg|_{\rho=0}^{\rho=4} \\
&= 2\pi \left(\frac{96}{3} - \frac{64}{3} \right) = \frac{64}{3}\pi.
\end{aligned}$$

(c) Ako primetimo da je površ data jednačinom $z = 9 - x^2 - y^2$ rotaciona, možemo zaključiti da je oblast V dobijena rotacijom oblasti

$$G = \{(y, z) \in \mathbb{R}^2 : z \leq 9 - y^2, -2 \leq y \leq 2\}$$

oko z-ose.





Zapreminu ćemo izračunati kao razliku zapremina oblasti V_1 i V_2, gde je

$$\begin{aligned} V_1 &= \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le 9, 0 \le z \le 9-x^2-y^2\} \\ V_2 &= \{(x,y,z) \in \mathbb{R}^3 : 4 \le x^2+y^2 \le 9, 0 \le z \le 5\}. \end{aligned}$$

Odatle je

$$\begin{aligned} \Delta V &= \Delta V_1 - \Delta V_2 = \iint\limits_{D_1} (9-x^2-y^2)dxdy - \iint\limits_{D_2} 5dxdy \\ &= \iint\limits_{D} (9-x^2-y^2)dxdy = \int\limits_0^{2\pi} d\varphi \int\limits_0^2 (9-\rho^2)\rho d\rho \\ &= 2\pi \cdot \left(\frac{9}{2}\rho^2 - \frac{1}{4}\rho^2\right)\Big|_0^2 = 34\pi. \end{aligned}$$

(d) Zapremina posmatrane oblasti jednaka je dvostrukoj zapremini dela koji leži iznad xy-ravni.

$$\begin{aligned} \Delta V &= 2\Delta V_1 \\ &= \iint\limits_D (2-x^2-y^2)dxdy \\ &= 2\int\limits_0^{2\pi} d\varphi \cdot \int\limits_0^1 (2-\rho^2)\rho d\rho = 3\pi. \end{aligned}$$