KOMPLEKSNI BROJEVI

* PO DEFINICIJI $i^2 = -1$, i – IMAGINARNA JEDINICA

* ALGEBARSKI OBLIK KOMPL. BROJA

$$\boxed{z = x + iy, \quad x, y \in \mathbb{R}}$$

$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$

SKUP KOMPL. BROJEVA

$Re\{z\} = x$, $Im\{z\} = y$

REALNI DEO, IMAGINARNI DEO

AKO JE $z_1 = x_1 + iy_1$ I $z_2 = x_2 + iy_2$, ONDA VAŽI:

$$\boxed{z_1 = z_2 \iff x_1 = x_2 \wedge y_1 = y_2}.$$

$$z = x + iy \rightarrow \boxed{\bar{z} \overset{df.}{=} x - iy}$$

$\bar{z}$ JE KONJUGOVANO-KOMPLEKSAN BROJ BROJU z.

* OSNOVNE OPERACIJE SA KOMPL. BROJEVIMA

- $z_1 \pm z_2 = (x_1 \pm x_2) + (y_1 \pm y_2)i$

- $z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_2)i$

- $$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2-(iy)^2} = \frac{x-iy}{x^2+y^2}$$
$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

- $z_1 : z_2 = \dfrac{z_1}{z_2} = z_1 \cdot \dfrac{1}{z_2}$

- $\overline{\overline{z}} = z$, $\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2$,

$\overline{z_1 \cdot z_2} = \overline{z}_1 \cdot \overline{z}_2$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$ $(z_2 \neq 0)$

$z \cdot \overline{z} = x^2 + y^2$, $\mathrm{Re}\{z\} = x = \frac{1}{2}(z + \overline{z})$,

$\mathrm{Im}\{z\} = y = \frac{1}{2i}(z - \overline{z})$

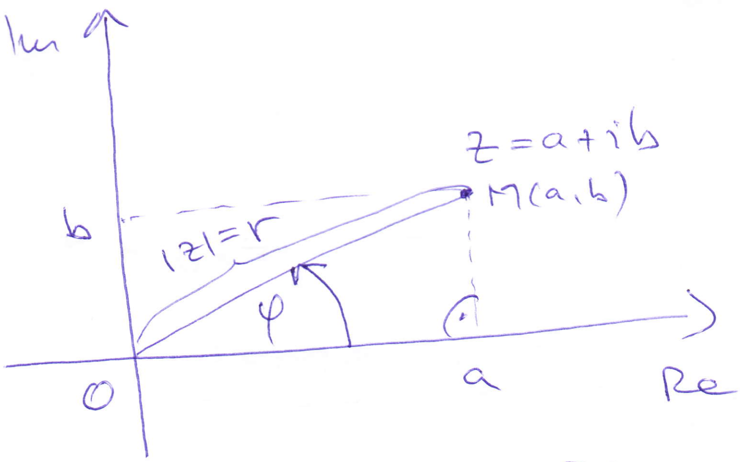
(*) GEOMETRIJSKA INTERPRETACIJA KOMPLEKSNOG BROJA

- ZA $z = x + iy$ BROJ $\boxed{|z| \stackrel{df.}{=} \sqrt{x^2+y^2}}$ SE NAZIVA MODUL ILI MODULO KOMPL. BROJA z.

- POSMATRAJMO KOMPLEKSNU RAVAN

x-OSA JE REALNA OSA
y-OSA JE IMAGINARNA OSA

$z = a + ib$



$|z| = r$ JE RASTOJANJE $|\overrightarrow{OM}|$.

φ JE ORIJENTISANI UGAO IZMEĐU POZITIVNOG DELA x-OSE I VEKTORA $\overrightarrow{OM}$. φ SE JOŠ NAZIVA <u>ARGUMENT</u> KOMPLEKSNOG BROJA z.

AKO JE $\underline{-\pi < \varphi \leq \pi}$ ONDA KAŽEMO DA JE φ <u>GLAVNI ARGUMENT</u> BROJA z, I PIŠEMO $\arg\{z\} = \varphi$.

SKUP SVIH ARGUMENATA

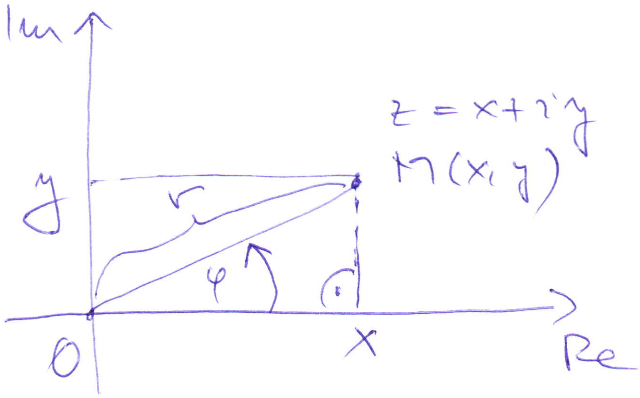
$$\mathrm{Arg}\{z\} = \{\arg\{z\} + 2k\pi \mid k\in\mathbb{Z}\}.$$

$a = r\cos\varphi$, $b = r\sin\varphi$

$a = r\cos(\varphi+2k\pi)$, $b = r\sin(\varphi+2k\pi)$, $k\in\mathbb{Z}$

(*) TRIGONOMETRIJSKI OBLIK KOMPLEKSNOG BROJA

$z = x + iy$



$\sin\varphi = \frac{y}{r} \Rightarrow y = r\sin\varphi$

$\cos\varphi = \frac{x}{r} \Rightarrow x = r\cos\varphi$

$z = x + iy = r\cos\varphi + i r\sin\varphi$

$= r(\cos\varphi + i\sin\varphi)$

$r = |z| = \sqrt{x^2+y^2}$

$\varphi = \arg\{z\}$

DAKLE, MOŽEMO PISATI DA JE

$$\boxed{z = r(\cos\varphi + i\sin\varphi)}$$

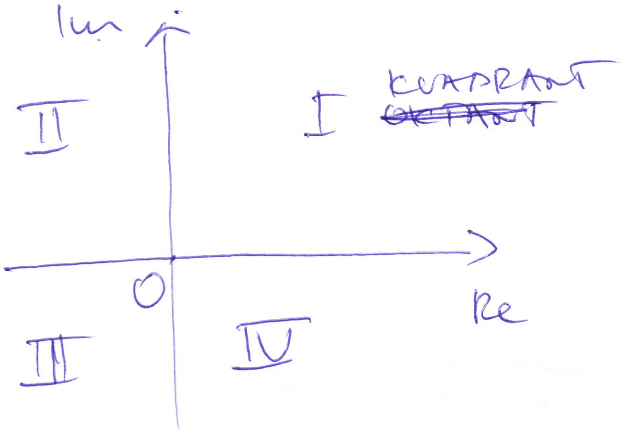
TRIGONOMETRIJSKI OBLIK KOMPL. BROJA z

$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$

$z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$

$z_1 = z_2 \Leftrightarrow r_1 = r_2 \wedge \mathrm{Arg}\{z_1\} = \mathrm{Arg}\{z_2\} + 2k\pi$

(*) PRETVARANJE IZ ALGEBARSKOG U TRIGONOMETRIJSKI OBLIK



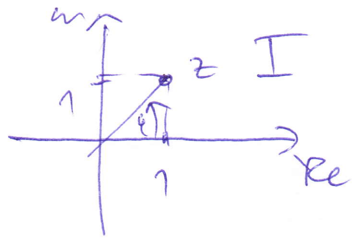
$$z = x + iy$$

$$-\pi < \arg\{z\} \leq \pi$$

I i IV	$\arg\{z\} = \operatorname{arctg} \frac{y}{x}$
II	$\arg\{z\} = \operatorname{arctg}\left(\frac{y}{x}\right) + \pi$
III	$\arg\{z\} = \operatorname{arctg}\left(\frac{y}{x}\right) - \pi$

• PRIMERI

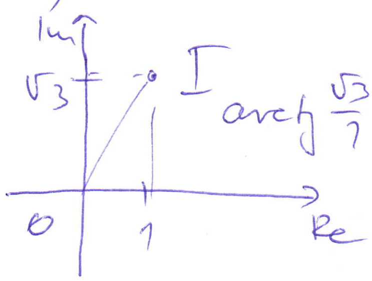
a) $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$



$|1 + i| = \sqrt{2}$

$\varphi = \frac{\pi}{4}$

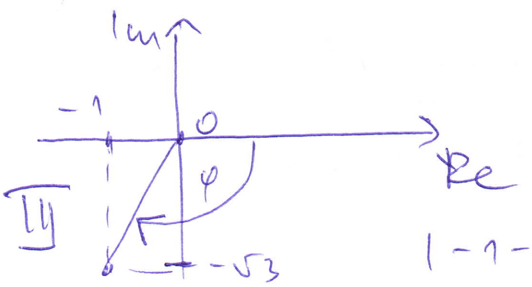
b) $1 + \sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$



$\operatorname{arctg}\frac{\sqrt{3}}{1} = \operatorname{arctg}\sqrt{3} = \frac{\pi}{3} = \varphi$

$|1 + \sqrt{3}| = \sqrt{1 + (\sqrt{3})^2} = 2$

c) $-1 - i\sqrt{3} = 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$



$\operatorname{arctg}\frac{-\sqrt{3}}{-1} = \operatorname{arctg}\sqrt{3} = \frac{\pi}{3}$

$|-1 - i\sqrt{3}| = 2$

$\varphi = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

* EKSPONENCIJALNI OBLIK KOMPL. BROJA

- AKO UVEDEMO OZNAKU $e^{\varphi i} \stackrel{\text{def.}}{=} \cos\varphi + i\sin\varphi$

TADA JE $z = r(\cos\varphi + i\sin\varphi) = r \cdot e^{\varphi i}$

$\boxed{z = r \cdot e^{\varphi i}}$ JE EKSPONENCIJALNI OBLIK (OJLEROV)

$r = |z|$, ~~$\varphi = \arg\{z\}$~~ $\varphi \in \operatorname{Arg}\{z\}$

- AKO JE $z_1 = r_1 e^{\varphi_1 i}$ I $z_2 = r_2 e^{\varphi_2 i}$, TADA JE

$$z_1 \cdot z_2 = r_1 \cdot r_2 \, e^{(\varphi_1 + \varphi_2)i}, \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{(\varphi_1 - \varphi_2)i}$$

- OJLEROVE (EULER-OVE) FORMULE

$$\cos\varphi + i\sin\varphi = e^{\varphi i}$$
$$\cos\varphi - i\sin\varphi = e^{-\varphi i} \quad (+) \quad (-)$$

$$2\cos\varphi = e^{\varphi i} + e^{-\varphi i}$$

$$\boxed{\cos\varphi = \frac{e^{\varphi i} + e^{-\varphi i}}{2}}$$

$$2i\sin\varphi = e^{\varphi i} - e^{-\varphi i}$$

$$\boxed{\sin\varphi = \frac{e^{\varphi i} - e^{-\varphi i}}{2i}}$$

"SINUS VOLI MINUS!"

POMOĆU OJLEROVIH FORMULA PROIZVOD TRIG. FUNKCIJA SE MOŽE PRETVORITI U ZBIR TRIG. FUNKCIJA.

PRIMER.

$$\cos\varphi \cdot \sin\varphi = \frac{e^{\varphi i} + e^{-\varphi i}}{2} \cdot \frac{e^{\varphi i} - e^{-\varphi i}}{2i} = \frac{1}{2} \cdot \frac{e^{2\varphi i} - e^{-2\varphi i}}{2i} =$$

$$= \frac{1}{2}\sin 2\varphi \quad \to \quad 2\cos\varphi\sin\varphi = \sin 2\varphi$$

* STEPENOVANJE KOMPL. BROJA

VAŽI DA JE $z_1 \cdot z_2 \cdots z_n = r_1 r_2 \cdots r_n \cdot (\cos(\varphi_1 + \cdots + \varphi_n) + i \sin(\varphi_1 + \cdots + \varphi_n))$

AKO JE $r_1 = r_2 = \cdots = r_n$ I $\varphi_1 = \varphi_2 = \cdots = \varphi_n$

TADA JE $\boxed{z^n = r^n e^{n\varphi i}}$, $n \in \mathbb{N}$

$\Downarrow$

$\cancel{r^n}(\cos\varphi + i\sin\varphi)^n = \cancel{r^n}(\cos n\varphi + i \sin n\varphi)$

$$\boxed{(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi}$$

MAVROV (MOIVRE-OV) OBRAZAC ZA STEPENOVANJE KOMP. BROJA

* KORENOVANJE KOMPL. BROJA

AKO JE $u \in \mathbb{C}$ DATO TRAŽIMO z TAKO DA BUDE $z^n = u$, $n \in \mathbb{N}$.

$z^n = u \to z = \sqrt[n]{u}$, $u = r(\cos\varphi + i\sin\varphi)$

r, φ DATE VELIČINE

NEKA JE $z = \rho(\cos\theta + i\sin\theta)$, $\rho, \theta = ?$

$z^n = \rho^n(\cos n\theta + i\sin n\theta) = r(\cos\varphi + i\sin\varphi) = u$

$\Leftrightarrow \rho^n = r \wedge n\theta = \varphi + 2k\pi$

$k \in \mathbb{Z}$

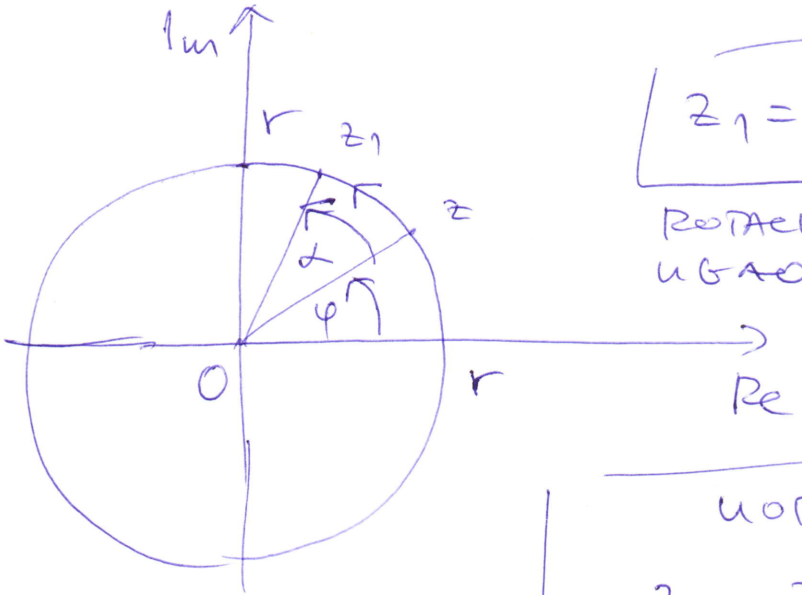
$\Leftrightarrow \rho = \sqrt[n]{r} \wedge \theta = \dfrac{\varphi + 2k\pi}{n}$, $k \in \mathbb{Z}$

ZA $k = n$ $\dfrac{\varphi + 2n\pi}{n} = \dfrac{\varphi}{n} + 2\pi \rightsquigarrow k = 0$ $\dfrac{\varphi}{n} + 0$

DAKLE $\boxed{z_k = \sqrt[n]{r}\left(\cos\dfrac{\varphi + 2k\pi}{n} + i\sin\dfrac{\varphi + 2k\pi}{n}\right), k = 0, 1, \ldots, n-1}$

(*) ROTACIJA

$$z = r\cdot e^{\varphi i} \xrightarrow{\text{ROT.}} z_1 = r\cdot e^{(\varphi+\alpha)i} = r e^{\varphi i}\cdot e^{\alpha i} = z\cdot e^{\alpha i}$$



$$\boxed{z_1 = z\cdot e^{\alpha i}}$$

ROTACIJA ZA UGAO α

UOPŠTENO

$$z_2 = z_0 + (z_1 - z_0)e^{\varphi i}$$

ROTACIJOM z_1 OKO z_0 ZA ORIJ. UGAO φ DOBIJA SE TAČKA z_2

AKO SU z_i REŠENJA OD $\sqrt[n]{z}$

TADA VAŽI

$$z_i = z_{i-1}\cdot e^{\frac{2\pi}{n}i}, \quad i = 1, \ldots, n-1$$