(*) MATRIČNI METOD REŠAVANJA LINEARNIH SISTEMA JEDNAČINA

POSMATRAJMO KVADRATNI SISTEM:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned} \qquad (n \times n).$$

NEKA JE A MATRICA SISTEMA, DEFINISANA KAO

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix},$$ VEKTOR NEPOZNATIH X

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{(n\times 1)},$$ A VEKTOR SLOBODNIH ČLANOVA $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{(n\times 1)}$

TADA POLAZNI SISTEM JEDNAČINA MOŽEMO NAPISATI U MATRIČNOM OBLIKU KAO $\boxed{A \cdot X = B}$. ZAISTA,

$$A \cdot X = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{bmatrix},$$

DAKLE,

$$A \cdot X = B \iff \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_n + \dots + a_{nn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

REŠAVANJEM MATRIČNE JEDNAČINE $A \cdot X = B$ MOŽEMO REŠITI POLAZNI LINEARNI SISTEM JEDN.:

$A^{-1} \cdot / \quad A \cdot X = B$, POD USLOVOM DA A^{-1} POSTOJI!
(TADA JE SISTEM ODREĐEN)

$$A^{-1}A \cdot X = A^{-1}B$$

$$E \cdot X = A^{-1}B$$

$$\boxed{X = A^{-1} \cdot B}$$

~~(TADA JES~~

PRIMER MATRIČNIM RAČUNOM REŠITI SISTEM LINEARNIH JEDNAČINA

$$2x - 3y + z = -1$$
$$x + y + z = 6$$
$$3x + y - 2z = -1.$$

REŠ.

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & -2 \end{bmatrix}, \quad \det(A) = -23 \neq 0 \implies A^{-1} \text{ POSTOJI!}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 6 \\ -1 \end{bmatrix} \quad \rightarrow \quad A \cdot X = B$$
$$X = A^{-1} \cdot B$$

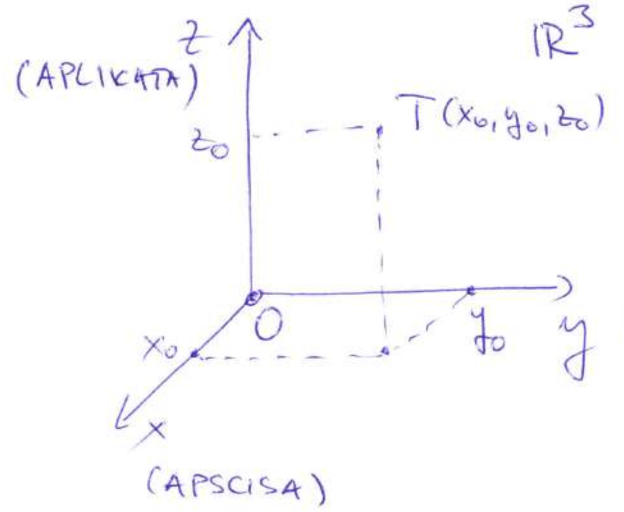
$$A^{-1} = -\frac{1}{23}\begin{bmatrix} -3 & -5 & -4 \\ 5 & -7 & -1 \\ -2 & -11 & 5 \end{bmatrix} \rightarrow X = \underbrace{-\frac{1}{23}\begin{bmatrix} -3 & -5 & -4 \\ 5 & -7 & -1 \\ -2 & -11 & 5 \end{bmatrix}}_{A^{-1}} \cdot \underbrace{\begin{bmatrix} -1 \\ 6 \\ -1 \end{bmatrix}}_{B} =$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightarrow x = 1, \; y = 2, \; z = 3$$

SLOBODNI VEKTORI U PROSTORU

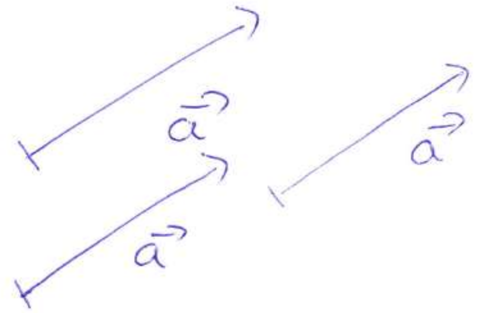
1 UVODNE DEFINICIJE

- KOORDINATNI SISTEM (DESNE ORIJENTACIJE)



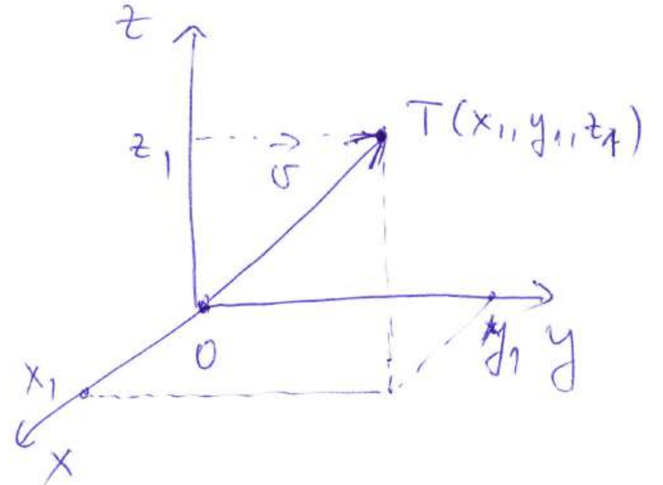
O - KOORDINATNI POČETAK

- VEKTOR JE VELIČINA (ORIJENTISANA DUŽ) KOJA IMA ODREĐENI PRAVAC, SMER I INTENZITET.



SLOBODAN VEKTOR PREDSTAVLJA SKUP MEĐUSOBNO PARALELNIH, PODUDARNIH I ISTO ORIJENTISANIH DUŽI.

VEKTOR NIJE FIKSIRAN!



- PREDSTAVLJANJE VEKTORA

$$\vec{v} = \overrightarrow{OT} = (x_1, y_1, z_1)$$

$\overrightarrow{OT} = \vec{r}_T \leftarrow$ VEKTOR POLOŽAJA TAČKE T

- $\vec{v}_1 = (x_1, y_1, z_1)$ i $\vec{v}_2 = (x_2, y_2, z_2)$

$$\vec{v}_1 = \vec{v}_2 \iff x_1 = x_2 \wedge y_1 = y_2 \wedge z_1 = z_2$$

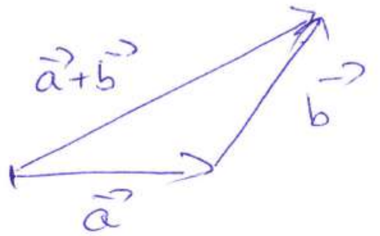
$\vec{0} = (0,0,0) \leftarrow$ NULA VEKTOR

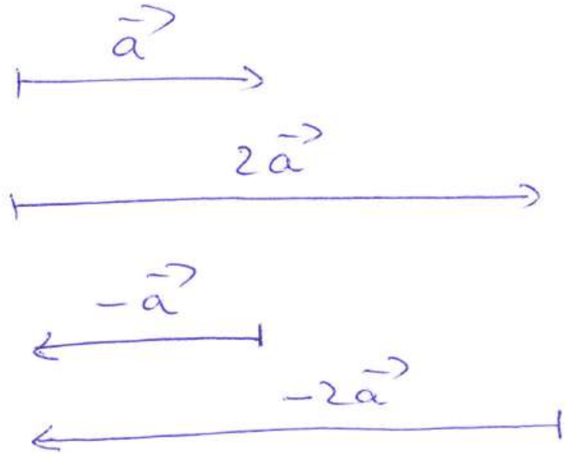
- INTENZITET VEKTORA

$$|\vec{v}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

- AKO JE $|\vec{v}| = 1 \Rightarrow \vec{v}$ JE JEDINIČNI VEKTOR.

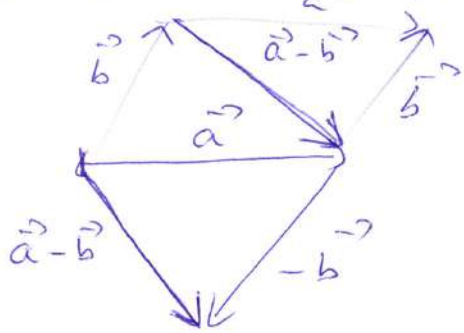
- SABIRANJE / ~~ODUZ~~ MNOŽENJE SKALAROM VEKTORA (GEOMETRIJSKO)





POČETAK DRUGOG VEKTORA SE NADOVEŽE NA KRAJ PRVOG, REZULTAT $\vec a + \vec b$

$$\vec a - \vec b = \vec a + (-\vec b)$$



$\lambda \vec a, \quad \lambda \in \mathbb{R}$

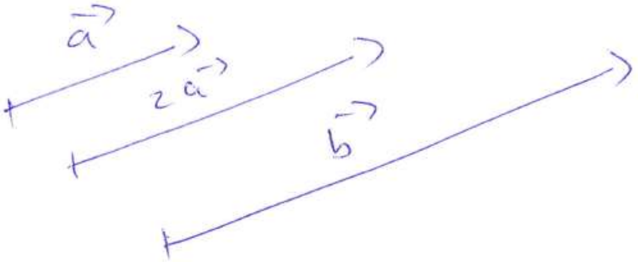
$\lambda \vec a$ — JE VEKTOR ISTOG PRAVCA KAO $\vec a$, A INTENZITET

$|\lambda \vec a| = \lambda \cdot |\vec a| \quad (\lambda > 0)$

$|\lambda \vec a| = -\lambda |\vec a| \quad (\lambda < 0)$

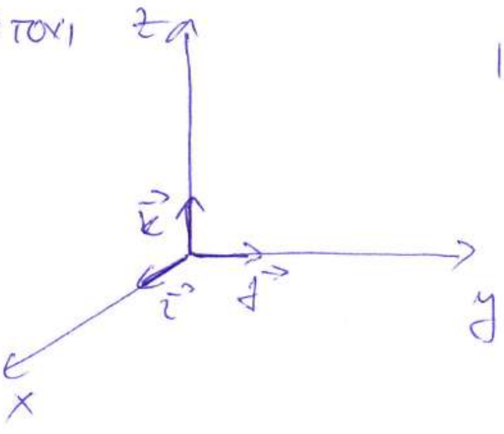
AKO JE $\lambda < 0 \Rightarrow$ DOBIJA SE VEKTOR SUPROTNE ORIJENTACIJE.

- VEKTORI KOJI IMAJU ISTI PRAVAC SU KOLINEARNI VEKTORI



$\vec a \parallel 2\vec a \parallel \vec b$

- ORTOVI



$|\vec i| = |\vec j| = |\vec k| = 1$

TADA ZA VEKTOR $\vec v = (x_1, y_1, z_1)$ MOŽEMO PISATI

$$\vec v = x_1 \vec i + y_1 \vec j + z_1 \vec k$$

- OPERACIJE PREKO KOORDINATA

(P5)

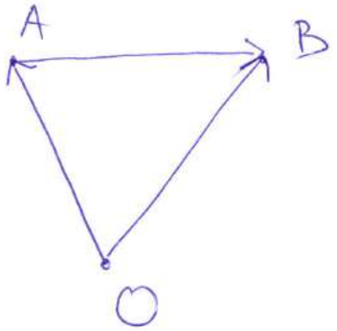
$\vec{v}_1 = (x_1, y_1, z_1)$, $\vec{v}_2 = (x_2, y_2, z_2)$

$\vec{v}_1 = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$, $\vec{v}_2 = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$

1) $\vec{v}_1 \pm \vec{v}_2 = (x_1 \pm x_2)\vec{i} + (y_1 \pm y_2)\vec{j} + (z_1 \pm z_2)\vec{k}$

2) $\lambda \cdot \vec{v}_1 = \lambda x_1 \vec{i} + \lambda y_1 \vec{j} + \lambda z_1 \vec{k}$, $\lambda \in \mathbb{R}$

- VEKTOR KOJI SPAJA DVE TAČKE



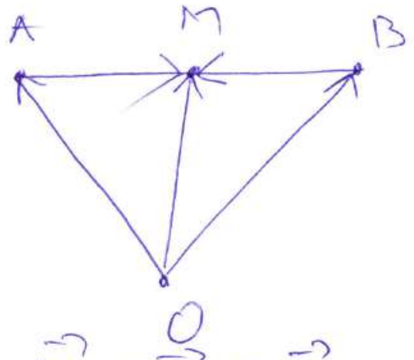
$\longrightarrow$ $\vec{OA} + \vec{AB} = \vec{OB}$

$\Downarrow$

$\boxed{\vec{AB} = \vec{OB} - \vec{OA}}$

$\vec{OB} = (x_2, y_2, z_2)$, $\vec{OA} = (x_1, y_1, z_1)$

$\vec{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$



$\vec{BM} + \vec{AM} = \vec{0}$

M - SREDINA DUŽI AB

TADA VAŽI

$$\begin{array}{l} \vec{OA} + \vec{AM} = \vec{OM} \\ \vec{OB} + \vec{BM} = \vec{OM} \quad + \\ \hline \vec{OA} + \vec{OB} = 2\vec{OM} \end{array}$$

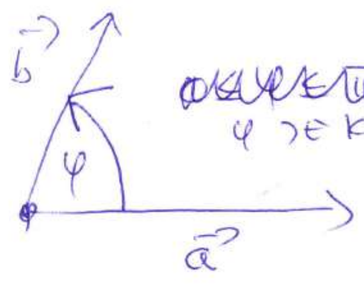
$\Downarrow$

$$\boxed{\vec{OM} = \frac{\vec{OA} + \vec{OB}}{2}} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, (z1+z2)/2 \right)$$

- UGAO IZMEĐU DVA VEKTORA

$\sphericalangle\ \varphi = \sphericalangle(\vec{a}, \vec{b})$

φ JE UGAO IZMEĐU VEKTORA $\vec{a}$ I $\vec{b}$



~~OKRENUTI~~ i orijentisan

φ JE KONVEKSAN UGAO

④ SKALARNI PROIZVOD DVA VEKTORA (P6)

~~SE~~

<u>DEF.</u> <u>SKALARNI PROIZVOD</u> VEKTORA $\vec{a}$ I $\vec{b}$

U OZNACI $\vec{a}\cdot\vec{b}$ DEFINIŠE SE

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\varphi,$$ GDE SE

φ UGAO IZMEĐU $\vec{a}$ I $\vec{b}$ $(\varphi=\measuredangle(\vec{a},\vec{b}))$.

- VAŽE SLEDEĆE OSOBINE

1) $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$,

2) $\vec{a}\cdot\vec{a}=|\vec{a}|^2$, $\cos 0^\circ=1$

3) $\vec{a}\cdot\vec{b}=0 \iff \vec{a}\perp\vec{b}$ (USLOV NORMALNOSTI)

4) $\forall\alpha\in\mathbb{R}\ \ \alpha(\vec{a}\cdot\vec{b})=(\alpha\vec{a})\cdot\vec{b}=\vec{a}\cdot(\alpha\vec{b})$

5) $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}+\vec{c}$

- NAČIN IZRAČUNAVANJA (PREKO KOORDINATA)

$$\underline{\vec{a}\cdot\vec{b}}=(x_1\vec{i}+y_1\vec{j}+z_1\vec{k})\cdot(x_2\vec{i}+y_2\vec{j}+z_2\vec{k})=$$

$$=x_1x_2\,\vec{i}\cdot\vec{i}+x_1y_2\,\overset{\to 0}{\cancel{\vec{i}\cdot\vec{j}}}+x_1z_2\,\overset{\to 0}{\cancel{\vec{i}\cdot\vec{k}}}+y_1x_2\,\overset{\to 0}{\cancel{\vec{i}\cdot\vec{j}}}+$$

$$+y_1y_2\,\vec{j}\cdot\vec{j}+y_1z_2\,\overset{\to 0}{\cancel{\vec{j}\cdot\vec{k}}}+z_1x_1\,\overset{\to 0}{\cancel{\vec{i}\cdot\vec{k}}}+z_1y_2\,\overset{\to 0}{\cancel{\vec{j}\cdot\vec{k}}}+z_1z_2\,\vec{k}\cdot\vec{k}$$

$$=\underline{x_1x_2+y_1\cdot y_2+z_1\cdot z_2}$$

- UGAO IZMEĐU DVA VEKTORA $\quad\cos\varphi=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$

(*) VEKTORSKI PROIZVOD DVA VEKTORA

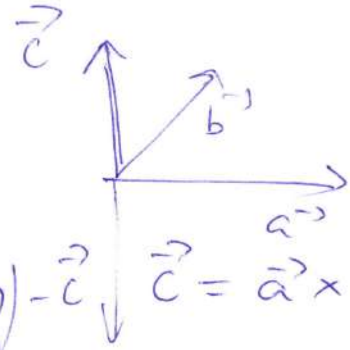
DEF. <u>VEKTORSKI PROIZVOD</u> VEKTORA $\vec{a}$ I $\vec{b}$ JE VEKTOR $\vec{a} \times \vec{b}$, ODREĐEN NA SLEDEĆI NAČIN

1) $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot |\sin \varphi|$, $\varphi \measuredangle (\vec{a}, \vec{b})$

2) $\vec{a} \times \vec{b} \perp \vec{a} \wedge \vec{a} \times \vec{b} \perp \vec{b}$

3) ~~VEKTOR~~ SMER OD $\vec{a} \times \vec{b}$ JE TAKAV DA VEKTORI $\vec{a}$, $\vec{b}$, I $\vec{a} \times \vec{b}$ ČINE DESNI SISTEM VEKTORA, TJ. SMER JE PREMA GORE AKO SE OD $\vec{a}$ KA $\vec{b}$ IDE U SMERU SUPROTNOM OD SMERA KAZALJKE NA SATU.

- OSOBINE



$\vec{c} = \vec{a} \times \vec{b}$

$-\vec{c} = \vec{b} \times \vec{a}$

1) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

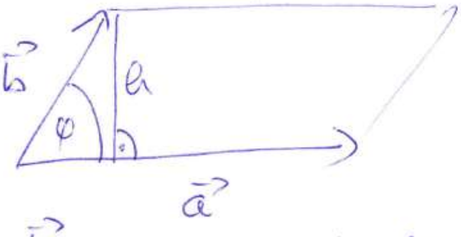
2) $\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a} \parallel \vec{b}$ (USLOV PARALELNOSTI ILI KOLINEARNOSTI)

3) $(\forall \alpha \in \mathbb{R})\ \alpha(\vec{a} \times \vec{b}) = (\alpha \vec{a}) \times \vec{b} = \vec{a} \times (\alpha \vec{b})$

4) $\vec{a} \times \vec{a} = 0$

5) $|\vec{a} \times \vec{b}| = P_{\square}$

POV. PARALELOGRAMA NAD VEKTORIMA $\vec{a}$ I $\vec{b}$



$$\sin \varphi = \frac{h}{|\vec{b}|}$$

$$h = |\vec{b}| \cdot \sin \varphi$$

$$P_{\square} = |\vec{a}| \cdot h = |\vec{a}| \cdot |\vec{b}| \cdot \sin \varphi = |\vec{a} \times \vec{b}|$$

- NAČIN IZRAČUNAVANJA

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \vec{i} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - \vec{j} \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + \vec{k} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

$\vec{a} = (x_1, y_1, z_1)$

$\vec{b} = (x_2, y_2, z_2)$

(*) MEŠOVITI PROIZVOD TRI VEKTORA

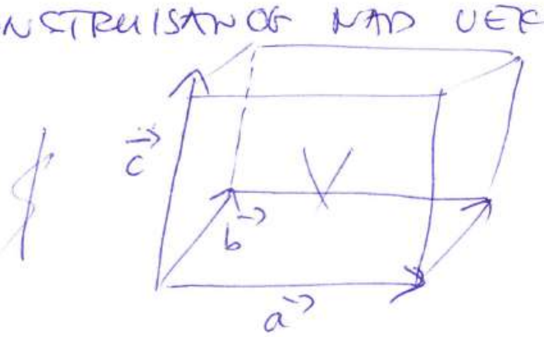
DEF. MEŠOVITI PROIZVOD VEKTORA $\vec{a}$, $\vec{b}$ I $\vec{c}$ U OZNACI $[\vec{a}, \vec{b}, \vec{c}]$, DEFINIŠE SE

$$[\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}).$$

- OSOBINE

1) $\vec{a}, \vec{b}, \vec{c}$ SU KOMPLANARNI (LEŽE U ISTOJ RAVNI)
$\iff [\vec{a}, \vec{b}, \vec{c}] = 0$ (USLOV KOMPLANARNOSTI)

2) $|[\vec{a}, \vec{b}, \vec{c}]| = V$, GDE JE V ZAPREMINA PARALELEPIPEDA KONSTRUISANOG NAD VEKTORIMA $\vec{a}, \vec{b}$ I $\vec{c}$



- NAČIN IZRAČUNAVANJA

$\vec{a} = (x_1, y_1, z_1)$, $\vec{b} = (x_2, y_2, z_2)$, $\vec{c} = (x_3, y_3, z_3)$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} =$$

$$= \vec{i} \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} - \vec{j} \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} + \vec{k} \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = x_1 \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} - y_1 \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} + z_1 \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$$

SKALARNI PROIZVOD

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$, DAKLE

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

VAŽI (CIKLIČKA PERMUTACIJA)

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$$