FURIJEOVA (FOURIER) I LAPLASOVA (LAPLACE) TRANSFORMACIJA

* FURIJEOVA TRANSFORMACIJA

ODREDIMO KOEFICIJENTE FURIJEOVOG REDA:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \quad , \quad x \in [-\pi, \pi]$$

$$a_n = ? \quad b_n = ?$$

$$\int_{-\pi}^{\pi} \ldots \, dx \; \Big/ \Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} a_0\,dx + \sum_{n=1}^{\infty} a_n \underbrace{\cancel{\int_{-\pi}^{\pi} \cos(nx)\,dx}}_{\to 0} + b_n \underbrace{\cancel{\int_{-\pi}^{\pi} \sin(nx)\,dx}}_{\to 0}$$

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2} a_0 x \Big|_{-\pi}^{\pi} = \frac{1}{2} a_0 (\pi + \pi) = \pi \cdot a_0$$

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx} \qquad n = 0$$

$$\int_{-\pi}^{\pi} \cos(nx) \; \Big/ \Rightarrow \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = a_n \int_{-\pi}^{\pi} \cos^2(nx)\,dx \underset{\substack{nx = t \\ dx = \frac{t}{n}}}{=}$$

$n \geq 1$

$$\left[\int_{-\pi}^{\pi} \cos(nx)\cdot\cos(mx)\,dx = 0 \;\; (n \neq m) \quad ; \quad \int_{-\pi}^{\pi} \cos(nx)\cdot\sin(mx)\,dx = 0 \right]$$

$$= \frac{1}{n} a_n \int_{-\pi \cdot n}^{\pi \cdot n} \frac{\cos 2t + 1}{2}\,dt = \frac{1}{2n}\cdot a_n \left[\frac{1}{2}\sin 2t + t\right]\Big|_{-\pi \cdot n}^{\pi \cdot n} =$$

$$= \frac{a_n}{2n}\left[\frac{1}{2}\cancel{\sin(2\pi \cdot n)}^{\to 0} + \pi n - \frac{1}{2}\cancel{\sin(-\pi \cdot n)}^{\to 0} + \pi \cdot n\right] = \frac{a_n}{\cancel{2n}} \cdot \cancel{2}\pi \cdot \cancel{n} = \pi \cdot a_n$$

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx} \qquad n = 1, 2, 3, \ldots$$

$$\int_{-\pi}^{\pi} \sin(nx) \quad \Big/ \quad \Rightarrow \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = b_n \int_{-\pi}^{\pi} \sin^2(nx)\,dx =$$

$nx = t$
$n\,dx = dt$

$n \geq 1$

$$\left[\int_{-\pi}^{\pi} \sin(nx)\,dx = 0, \quad \int_{-\pi}^{\pi} \sin(nx)\cdot\cos(mx)\,dx = 0, \quad \int_{-\pi}^{\pi} \sin(nx)\cdot\sin(mx)\,dx = 0 \right]$$

$n \neq m$

$$= b_n \int_{-\pi\cdot n}^{\pi\cdot n} \sin^2 t \cdot \frac{dt}{n} = \frac{b_n}{n} \int_{-\pi\cdot n}^{\pi\cdot n} \frac{1-\cos 2t}{2}\,dt = \dots = b_n \cdot \pi$$

$$\boxed{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx} \qquad n = 1, 2, \dots$$

RAZVOJ FUNKCIJE $f(x)$ U FURIJEOV RED NA INT. $[-\pi, \pi]$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$n \geq 1$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

ČESTO SE KORISTI ZAPIS U KOMPLEKSNOJ FORMI:

$$F(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx, \quad n \in \mathbb{Z} \leftarrow$$ DIREKTNA FURIJEOVA TRANSF.

TADA JE (MOŽE SE POKAZATI)

$$\sum_{n=-\infty}^{\infty} F(n) e^{inx} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = f(x)$$

$x \in [-\pi, \pi]$

$\Downarrow$ (GENERALIZACIJA)

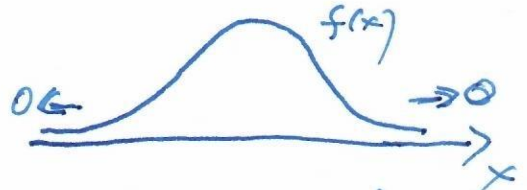
$$F(\omega) = \int_{-\infty}^{\infty} f(x) \cdot e^{-i\omega x}\,dx, \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\, e^{i\omega x}\,dx$$

DIREKTNA FUR. TRANSF. INVERZNA FUR. TRANSF.

FURIJEOVA TRANSFORMACIJA

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\cdot e^{-i\omega x}\,dx$$

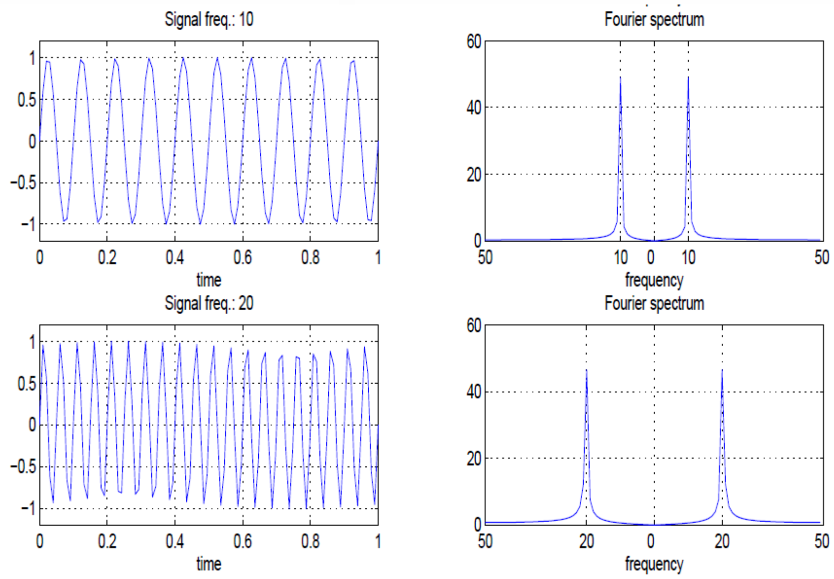
USLOV KONVERGENCIJE $\lim_{x\to\pm\infty} f^{(n)}(x) = 0, \quad n=0,1,2,\ldots$

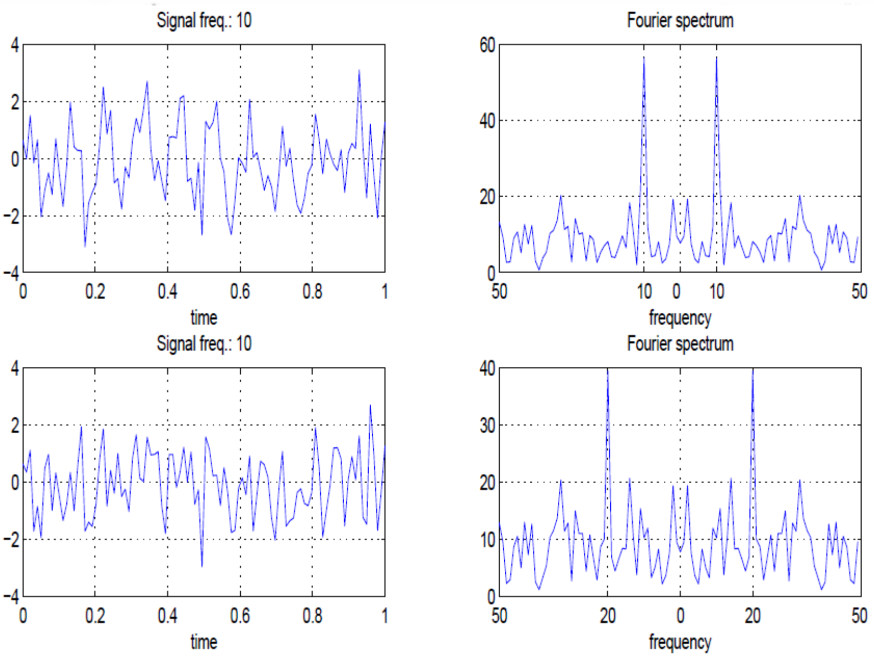


$$F(\omega) = F_R(\omega) + i\,F_I(\omega) \in \mathbb{C}, \quad \omega \text{ – FREKVENCIJA}$$

$|F(\omega)|$ – ~~FR~~ SPEKTAR (SPECTRUM) ; $\arg(F(\omega))$ – FAZA (PHASE)

PRIMERI





INVERZNA FURIJEOVA TRANSFORMACIJA

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, dx$$, ω – FREKVENCIJA

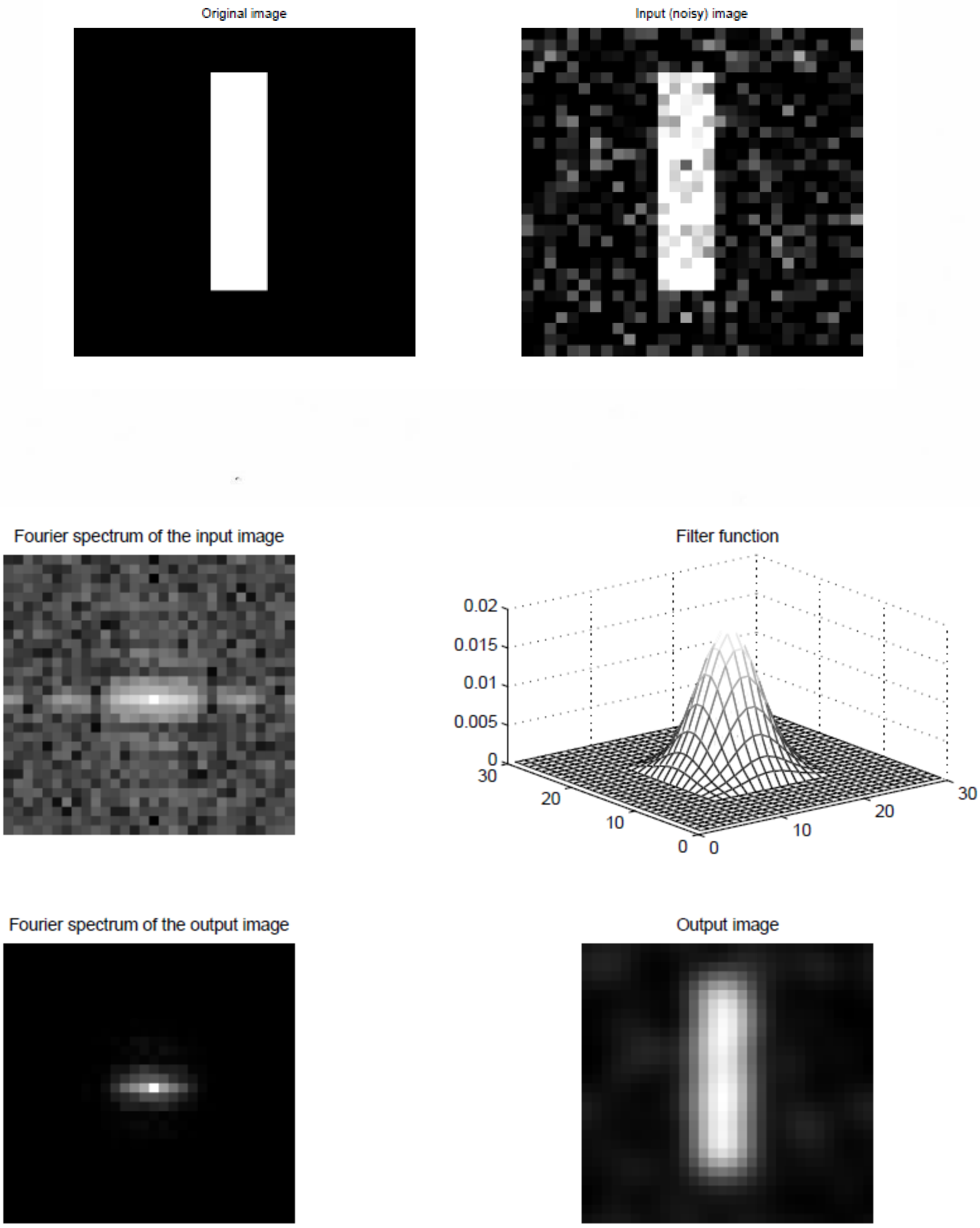
(u 2D):

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i 2\pi (u\cdot x + v\cdot y)}\, dx\, dy$$ ← DIREKTNA FUR. T.

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\cdot e^{i 2\pi (u\cdot x + v\cdot y)}\, du\, dv$$ ← INV. FUR. TRANSF.

u, v – FREKVENCIJE

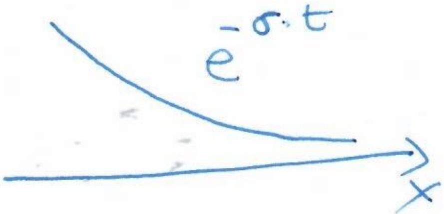
PRIMERI



✳ LAPLASOVA TRANSFORMACIJA

$$H(t) \overset{\text{def.}}{=} \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}$$

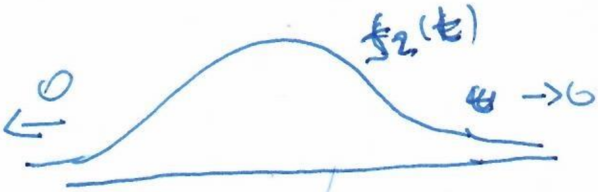
HEVISAJD FUNKCIJA



POSMATRAJMO NOVU FUNKCIJU $f_2(t) = f(t) \cdot e^{-\sigma \cdot t} \cdot H(t)$,
TADA VAŽI DA JE

$$f_2(t) = \begin{cases} 0, & t<0 \\ f(t) e^{-\sigma \cdot t}, & t \geq 0 \end{cases}$$

$f_2(t)$ ZADOVOLJAVA USLOV KONVERGENCIJE ~~TRANSF.~~ FURIJEOVE ~~REDA~~ ZA ŠIROKU KLASU FUNKCIJA $f(t)$.



FURIJEOVA TRANSFORMACIJA ZA $f_2(t)$

$$F_2(\omega) = \int_{-\infty}^{\infty} f_2(t) \cdot e^{-i\omega t} dt = \int_{-\infty}^{\infty} f(t) e^{-\sigma \cdot t} \cdot \underbrace{e^{-i\omega t}}_{\cdot H(t)} dt =$$

$$= \int_{0}^{\infty} f(t) \cdot e^{-(\sigma + i\omega)\cdot t} dt = \int_{0}^{\infty} f(t) e^{-s \cdot t} dt = \mathcal{L}\{f\}$$

$s = \sigma + i\omega \in \mathbb{C}$

LAPLASOVA TRANSF.

DAKLE, FURIJEOVA TRANSFORMACIJA OD $f_2(t)$ JE LAPLASOVA TRANSFORMACIJA OD $f(t)$.

$$\mathcal{F}(f_2) = \mathcal{L}\{f\}$$

$$\int_{-\infty}^{\infty} \underbrace{f(t) \cdot H(t) \cdot e^{-\sigma t}}_{f_2(t)} \cdot e^{-i\omega t} \cdot dt = \int_{0}^{\infty} f(t) e^{-s \cdot t} dt$$

$s = \sigma + i\omega$, $\sigma, \omega \in \mathbb{R}$

<u>LAPLASOVA TRANSFORMACIJA JE GENERALIZOVANA FURIJEOVA TRANSFORMACIJA.</u>