LAPLASOVE TRANSFORMACIJE

(*) DEFINICIJA I OSOBINE

DEF. (LAPLASOVA TRANSFORMACIJA)
NEKA JE $f(t)$ REALNA FUNKCIJA, NEPREKIDNA PO DELOVIMA NA INTERVALU $(0,+\infty)$. AKO JE NESVOJSTVENI INTEGRAL

$$\int_0^{+\infty} f(t)e^{-st}\,dt$$

KONVERGENTAN, ONDA ON DEFINIŠE FUNKCIJU $F(s)$, TJ.

$$F(s) \overset{df.}{=} \int_0^{+\infty} f(t)e^{-s\cdot t}\,dt .$$

FUNKCIJU $f(t)$ NAZIVAMO ORIGINAL, A FUNKCIJU $F(s)$ SLIKA. TRANSFORMACIJA KOJA PRESLIKAVA $f(t)$ U $F(s)$ SE ZOVE LAPLASOVA TRANSFORMACIJA, PIŠEMO

$$\boxed{\mathcal{L}\{f(t)\} = F(s).}$$

JASNO, $\mathcal{L}\{f(t)\} = \int_0^{+\infty} f(t)e^{-st}\,dt$.

DEF. (EKSPONENCIJALNI RED) AKO ZA FUNKCIJU $f(t)$ POSTOJE KONSTANTE $M>0, T>0, K \in \mathbb{R}$ TAKVE DA JE

$$|f(t)| \le M\cdot e^{K\cdot t}, \quad \text{ZA } t \ge T$$

ONDA ZA FUNKCIJU $f(t)$ KAŽEMO DA JE EKSPONENCIJALNOG REDA. NAJMANJI OD SVIH BROJEVA K ZA KOJE JE NEJEDNAKOST ZADOVOLJENA, NAZIVAMO RED EKSPONENCIJALNOG RASTA.

• (EGZISTENCIJA LAP. TRANSF.) MOŽE SE POKAZATI (P2)
DA NEPREKIDNA FUNKCIJA $f(t)$, KOJA JE EKSPONENCIJALNOG REDA, IMA LAPLASOVU TRANSFORMACIJU, TJ. POSTOJI $F(s)$:

$$\mathcal{L}\{f(t)\} = F(s) \text{ za } \mathrm{Re}\{s\} > K.$$

• OSOBINE LAPLASOVE TRANSFORMACIJE

1) $\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$

DOKAZ: $$\mathcal{L}\{f+g\} = \int_0^{+\infty} (f(t)+g(t))e^{-st}\,dt = \int_0^{+\infty} f(t)e^{-st}\,dt + \int_0^{+\infty} g(t)e^{-st}\,dt = \mathcal{L}\{f\} + \mathcal{L}\{g\}$$

2) $\mathcal{L}\{c\cdot f(t)\} = c\,\mathcal{L}\{f(t)\}$, $c\in\mathbb{R}$

DOKAZ: $$\mathcal{L}\{cf(t)\} = \cancel{c\,\mathcal{L}\{f(t)\}} \int_0^{+\infty} c\cdot f(t)e^{-st}\,dt = c\int_0^{+\infty} f(t)e^{-st}\,dt$$

• PRIMER. ODREDITI PO DEFINICIJI LAPLASOVU ~~TRANSFORM~~ TRANSFORMACIJE SLEDEĆIH FUNKCIJA.

a) $f(t)=0$

$$\mathcal{L}\{0\} = \int_0^{+\infty} 0\cdot e^{-st}\,dt = \int_0^{+\infty} 0 = 0$$

$$\boxed{\mathcal{L}\{0\}=0}$$

b) $f(t)=c$, $c\in\mathbb{R}$

$$\mathcal{L}\{c\} = \int_0^{+\infty} c\cdot e^{-st}\,dt = c\cdot\int_0^{+\infty} e^{-st}\,dt = c\cdot\lim_{T\to+\infty}\int_0^{T} e^{-st}\,dt =$$

$$= c\cdot\lim_{T\to+\infty} -\frac{1}{s}e^{-st}\Big|_0^T = +\frac{c}{s}\Big(\underbrace{\lim_{T\to\infty} -e^{-sT}}_{\substack{0\\ s>0}} + e^0\Big) = \frac{c}{s}$$

$\boxed{\mathcal{L}\{c\} = \frac{c}{s}}, \quad s>0 \quad \Rightarrow \boxed{\mathcal{L}\{1\} = \frac{1}{s}}$
$c=1$

c) $f(t) = e^{at}, \quad a \in \mathbb{R}$

$$\mathcal{L}\{e^{at}\} = \int_0^{+\infty} e^{at} \cdot e^{-st}\,dt = \int_0^{+\infty} e^{(a-s)t}\,dt = \lim_{T\to+\infty} \int_0^T e^{(a-s)t}\,dt =$$

$$\int e^{(a-s)t}\,dt = \int e^z \cdot \frac{dz}{a-s} = \frac{1}{a-s}\int e^z\,dz = \frac{1}{a-s} \cdot e^z = \frac{1}{a-s} \cdot e^{(a-s)t}$$

$(a-s) \cdot t = z$

$(a-s)dt = dz$

$dt = \frac{dz}{a-s}$

$$= \lim_{T\to+\infty} \frac{1}{a-s} \cdot e^{(a-s)\cdot t}\Big|_0^T = \frac{1}{a-s} \lim_{T\to+\infty}\left(\cancel{e^{(a-s)\cdot T}} - e^0\right) =$$

$a-s<0 \Leftrightarrow -s<-a$

$\Leftrightarrow s>a$

$$= \frac{-1}{a-s} = \frac{1}{s-a}, \quad s>a$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s>a}$$

~~PRIMER.~~

T (PRIGUŠIVANJE) NEKA JE $\mathcal{L}\{f\} = F(s)$. TADA ZA PROIZVOLJAN REALAN BROJ α VAŽI

$$\mathcal{L}\{e^{\alpha t} f(t)\} = F(s-\alpha).$$

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POKAZ

$$\mathcal{L}\{e^{\alpha t} f(t)\} = \int_0^{+\infty} f(t)\, e^{\alpha t} \cdot e^{-st}\, dt = \int_0^{+\infty} f(t) \cdot e^{-(s-\alpha)t}\, dt = \quad (s-\alpha = z)$$

$$= F(s-\alpha), \quad s > \alpha.$$

(T) (SLIČNOST) NEKA JE $\mathcal{L}\{f(t)\} = F(s)$. TADA ZA PROIZVOLJAN REALAN BROJ $a>0$ VAŽI

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right).$$

POKAZ

$$\mathcal{L}\{f(at)\} = \int_0^{+\infty} f(at) \cdot e^{-st}\, dt = \int_0^{+\infty} f(u)\, e^{-s \cdot \frac{u}{a}} \cdot \frac{1}{a}\, du =$$

SMENA: $at = u$ $\quad t \to 0 \Rightarrow u \to 0$

$a\,dt = du$ $\quad t \to +\infty \Rightarrow u \to +\infty$

$dt = \frac{1}{a} du$ $\quad t = \frac{u}{a}$

$$= \frac{1}{a} \int_0^{+\infty} f(u)\, e^{-s \cdot \frac{u}{a}}\, du = \frac{1}{a} \int_0^{+\infty} f(u) \cdot e^{-\frac{s}{a} \cdot u} \cdot du = \quad \left(\frac{s}{a} = z\right)$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right), \quad s > 0$$

PRIMER ODREDITI LAPLASOVU TRANSFORMACIJU SLEDEĆIH FUNKCIJA

a) $f(t) = \cos(at)$, $a \in \mathbb{R}$

NA OSNOVU OJLEROVE FORMULE VAŽI $\cos(at) = \dfrac{e^{iat} + e^{-iat}}{2}$

$$\mathcal{L}\{\cos(at)\} = \mathcal{L}\left\{\frac{e^{iat}+e^{-iat}}{2}\right\} =$$

$$= \frac{1}{2}\mathcal{L}\{e^{iat}+e^{-iat}\} = \frac{1}{2}\mathcal{L}\{e^{iat}\} + \frac{1}{2}\mathcal{L}\{e^{-iat}\} =$$

$$= \frac{1}{2}\frac{1}{s-ia} + \frac{1}{2}\frac{1}{s+ia} = \frac{s}{s^2+a^2}$$

$$\boxed{\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}}$$

b) $f(t) = \sin(at)$, $a \in \mathbb{R}$

$$\sin(at) = \frac{e^{iat}-e^{-iat}}{2i}$$

$$\mathcal{L}\{\sin(at)\} = \frac{1}{2i}\mathcal{L}\{e^{iat}\} - \frac{1}{2i}\mathcal{L}\{e^{-iat}\} =$$

$$= \frac{1}{2i}\frac{1}{s-ia} - \frac{1}{2i}\frac{1}{s+ia} = \frac{a}{s^2+a^2}$$

$$\boxed{\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}}$$

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Osobine Laplasove transformacije

1. **Linearnost** $\mathcal{L}[\alpha f(t)+\beta g(t)]=\alpha F(s)+\beta G(s)$.
2. **Sličnost** $\mathcal{L}[f(at)]=\frac{1}{a}F\left(\frac{s}{a}\right)$, $a>0$.
3. **Prigušivanje** $\mathcal{L}[e^{\alpha t}f(t)]=F(s-\alpha)$.
4. **Kašnjenje** $\mathcal{L}[f(t-a)]=e^{-as}F(s)$, $a>0$.
5. **Kašnjenje po parametru** Ako je $\mathcal{L}[f(t,x)]=F(s,x)$, tada je
$$\mathcal{L}\left[\frac{\partial f(t,x)}{\partial x}\right]=\frac{\partial F(s,x)}{\partial x}.$$
6. **Izvod originala** $\mathcal{L}[f'(t)]=sF(s)-f(0^+)$,
$$\mathcal{L}[f^{(n)}(t)]=s^nF(s)-s^{n-1}f(0)^+-...-f^{(n-1)}(0^+).$$
7. **Izvod slike** $\mathcal{L}[-tf(t)]=F'(s)$.
8. **Integracija originala** $\mathcal{L}\left[\int_0^t f(u)du\right]=\frac{F(s)}{s}$.
9. **Integracija slike** $\mathcal{L}\left[\frac{f(t)}{t}\right]=\int_s^\infty F(u)du$.
10. **Proizvod originala** $\mathcal{L}[f(t)g(t)]=(F*G)(s)$.
11. **Proizvod slika** $\mathcal{L}[(f*g)(t)]=\mathcal{L}[\int\limits_0^t f(u)g(t-u)du]=F(s)G(s)$.

$$\boxed{\begin{aligned}\mathcal{L}\{y'\}&=s\cdot\mathcal{L}\{y\}-y(0)\\ \mathcal{L}\{y''\}&=s^2\mathcal{L}\{y\}-s\cdot y(0)-y'(0)\end{aligned}}$$

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Tablica Laplasovih transformacija

	$F(s)$	$f(t)$
1.	$\frac{1}{s}$	1
2.	$\frac{1}{s^2}$	t
3.	$\frac{1}{s^n}$, $n \in \mathbb{N}$	$\frac{t^{n-1}}{(n-1)!}$
4.	$\frac{1}{s^n}$, $n > 0$	$\frac{t^{n-1}}{\Gamma(n)}$
5.	$\frac{1}{s-a}$	e^{at}
6.	$\frac{1}{(s-a)^n}$, $n \in \mathbb{N}$	$\frac{t^{n-1}e^{at}}{(n-1)!}$
7.	$\frac{1}{(s-a)^n}$, $n > 0$	$\frac{t^{n-1}e^{at}}{\Gamma(n)}$
8.	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
9.	$\frac{s}{s^2+a^2}$	$\cos at$
10	$\frac{1}{(s-b)^2+a^2}$	$\frac{e^{bt}\sin at}{a}$
11.	$\frac{s-b}{(s-b)^2+a^2}$	$e^{bt}\cos at$
12.	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
13.	$\frac{s}{s^2-a^2}$	$\cosh at$
14.	$\frac{1}{(s-b)^2-a^2}$	$\frac{e^{bt}\sinh at}{a}$
15.	$\frac{s-b}{(s-b)^2-a^2}$	$e^{bt}\cosh at$
	$\sqrt{\frac{\pi}{s}}$	$t^{-\frac{1}{2}}$
16.	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t \cdot \cos at$
17.	$\frac{2as}{(s^2+a^2)^2}$	$t \cdot \sin at$

PRIMER IZRAČUNATI LAPLASOVE TRANSFORMACIJE

a) $\mathcal{L}\{3t + e^{2t}\} = \mathcal{L}\{3t\} + \mathcal{L}\{e^{2t}\} =$

$= 3\,\mathcal{L}\{t\} + \mathcal{L}\{e^{2t}\} = 3 \cdot \underset{(2)}{\frac{1}{s^2}} + \underset{(5)}{\frac{1}{s-2}} = \frac{3}{s^2} + \frac{1}{s-2}$

b) $\mathcal{L}\{(e^{3t} - t)\sin 3t\} = \underset{(10)}{\mathcal{L}\{e^{3t}\sin 3t\}} - \underset{(17)}{\mathcal{L}\{t \sin 3t\}} =$

$= \frac{3}{(s-3)^2 + 3^2} - \frac{2 \cdot 3\, s}{(s^2+a^2)^2} = \frac{3}{(s-3)^2+9} - \frac{6s}{(s^2+9)^2}$

(*) INVERZNA LAPLASOVA TRANSFORMACIJA

$$\mathcal{L}\{f\} = F \quad \rightsquigarrow \quad \mathcal{L}^{-1}\{F\} = f$$

$\mathcal{L}^{-1}$ – INV. LAP. TRANSFORMACIJA

- OSOBINE $\mathcal{L}^{-1}$

1) $\mathcal{L}^{-1}\{F(s) + G(s)\} = \mathcal{L}^{-1}\{F(s)\} + \mathcal{L}^{-1}\{G(s)\} =$

$= f(t) + g(t)$

2) $\mathcal{L}^{-1}\{c \cdot F(s)\} = c \cdot \mathcal{L}^{-1}\{F(s)\} = c \cdot f(t), \quad c \in \mathbb{R}$

PRIMER. ODREDITI INVERZNU LAP. TRANSFORMACIJU ZA

a) $F(s) = \frac{1}{s}$

$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{1}{s}\} = 1$
(1)

b) $F(s) = \frac{1}{s-2}$

$\mathcal{L}^{-1}\{\frac{1}{s-2}\} = e^{2t}$
(5)

c) $F(s) = \dfrac{s}{(s-9)^2}$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-9)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s-3^2)^2}\right\} = \underset{(5)}{\mathcal{L}^{-1}\left\{\frac{1}{s-9}\right\}} + \underset{(6)}{\mathcal{L}^{-1}\left\{\frac{9}{(s-9)^2}\right\}} =$$

$$\frac{s}{(s-3^2)^2} = \frac{A}{s-3^2} + \frac{B}{(s-3^2)^2} = \frac{1}{s-9} + \frac{9}{(s-9)^2}$$

$$= e^{9t} + 9 \cdot t e^{9t} = e^{9t}(9t+1)$$

d) $\cancel{F(s) = \dfrac{2s+3}{s^2+4s+13}}$ $\quad F(s) = \dfrac{1}{s(s+3)}$

$\cancel{\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{}$

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} \quad / \cdot (s(s+3))$$

$$1 = A(s+3) + B \cdot s$$

$$1 = s(A+B) + 3A$$

$$\left.\begin{array}{l} A+B=0 \\ 3A=1 \end{array}\right. \iff \begin{array}{l} B = -\frac{1}{3} \\ A = \frac{1}{3} \end{array} \implies \frac{1}{s(s+3)} = \frac{1}{3 \cdot s} - \frac{1}{3(s+3)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+3)}\right\} = \frac{1}{3}\underset{(1)}{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}} - \frac{1}{3}\underset{(5)}{\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}} =$$

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$$s+3 = s-\underbrace{(-3)}_{a}$$

$$= \frac{1}{3} \cdot 1 - \frac{1}{3} \cdot e^{-3t} = \frac{1}{3}\left(1 - e^{-3t}\right)$$

(*) PRIMENA LAPLASOVE TRANSFORMACIJE NA REŠAVANJE DIFERENCIJALNIH JEDNAČINA

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IDEJA

1) PRIMENA DIREKTNE LAP. TRANSF. NA DATU DIFERENCIJALNU JEDNAČINU

AKO JE $y(t)$ - NEPOZNATA FUNKCIJA $\rightsquigarrow \mathcal{L}\{y(t)\} = Y(s)$

2) REŠAVANJE DOBIJENE ALGEBARSKE JEDNAČINE I ODREĐIVANJE (IZRAŽAVANJE) $Y(s) =$

3) PRIMENA INVERZNE LAP. TRANSF. $\mathcal{L}^{-1}\{Y\} = y(t)$

I ODREĐIVANJE $y(t)$

PRIMER PRIMENOM LAPLASOVE TRANSFORMACIJE REŠITI DIFERENCIJALNU JEDNAČINU

$$y''(t) + y(t) = e^{-t}$$

UZ POČETNE USLOVE $y(0) = 1$, $y'(0) = 2$.

REŠ. PRIMENIMO LAP. TRANSF. NA DATU DIF. JEDNAČINU

$$\mathcal{L}\{y''(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{e^{-t}\}$$

$$s^2\mathcal{L}\{y\} - s\cdot y(0) - y'(0) + \mathcal{L}\{y\} = \frac{1}{s+1}$$

$$(s)$$

$$s^2 Y - s\cdot 1 - 2 + Y = \frac{1}{s+1}$$

$$Y(s^2+1) = \frac{1}{s+1} + s + 2 = \frac{s+1+s^2+s+2}{s+1}$$

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$$Y = \frac{s^2+3s+3}{(s+1)(s^2+1)} = \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{2}\,\frac{s+5}{(s^2+1)}$$

$$\frac{s^2+3s+3}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$\vdots$$

$$A = \frac{1}{2}\ ,\ B = \frac{1}{2}\ ,\ C = \frac{5}{2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{s+5}{s^2+1}\right\} =$$

(5)

$$= \frac{1}{2}\cdot e^{-t} + \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{5}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} =$$

(9)

$$= \frac{1}{2}\,e^{-t} + \frac{1}{2}\cdot\cos t + \frac{5}{2}\sin t$$