(P1)

(*) OPERATOR KONVOLUCIJE

POSMATRAJMO LAPLASOVE TRANSFORMACIJE FUNKCIJA $f(t)$ i $g(t)$, $t \in \mathbb{R}$:

$$F(s) = \mathcal{L}\{f\} = \int_0^{+\infty} f(u) e^{-s \cdot u} du \quad i$$

$$G(s) = \mathcal{L}\{g\} = \int_0^{+\infty} g(v) e^{-s \cdot v} dv.$$

TADA MOŽEMO PISATI

$$F(s) \cdot G(s) = \int_0^{+\infty} f(u) e^{-s \cdot u} du \cdot \int_0^{+\infty} g(v) e^{-s \cdot v} dv =$$

$$= \int_0^{+\infty} \int_0^{+\infty} f(u) g(v) e^{-s(u+v)} du dv,$$

JER JE DVOSTRUKI INTEGRAL SEPARABILAN, TJ. NJEGOVA PODINTEGRALNA FUNKCIJA JE PROIZVOD DVE FUNKCIJE OD PO JEDNE PROMENLJIVE. UVEDIMO SMENU U DVOSTR. INT.:

dudt

$$\begin{cases} u = u(u,t) = u \\ v = v(u,t) = t - u \end{cases} \longrightarrow du dv = |J| = 1 \cdot du dt$$

$$J = \begin{vmatrix} u'_u & u'_t \\ v'_u & v'_t \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$\left| \frac{\partial(u,v)}{\partial(u,t)} \right|$$

$$u + v = u + t - u = t$$

TADA JE $\displaystyle \int_0^{+\infty} \int_0^{+\infty} f(u) g(v) e^{-s(u+v)} du dv = \int_0^{+\infty} dt \int_0^{t} f(u) g(t-u) e^{-s \cdot t} du$

$$= \int_0^{+\infty} \left(\int_0^{t} f(u) g(t-u) du \right) e^{-st} dt$$

ZA GRANICE INTEGRACIJE VAŽI:

$0 \leq u \leq +\infty$

$0 \leq v \leq +\infty$

SMENA

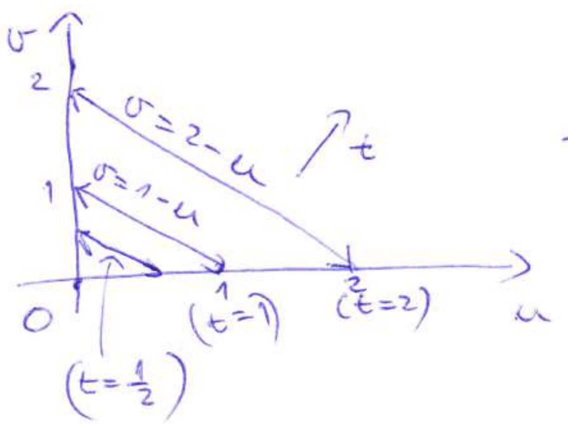
$$\begin{cases} v = t - u \\ u = u \end{cases}$$

STARE PROM (u, v)

$u = u(u, t)$

$v = v(u, t)$

(u, t) NOVE PROM.



GRANICE U ODNOSU NA NOVE PROMENLJIVE

$0 \leq t \leq +\infty$

$0 \leq u \leq t$

DAKLE, DOBILI SMO JEDNAKOST

$$F(s) \cdot G(s) = \int_0^{+\infty} \left(\int_0^t f(u)\, g(t-u)\, du \right) \cdot e^{-s \cdot t}\, dt .$$

<u>OPERATOR KONVOLUCIJE</u> ZA DVE FUNKCIJE $f(t)$ I $g(t)$ DEFINIŠEMO NA SLEDEĆI NAČIN

$$\boxed{f * g(t) \overset{\text{df.}}{=} \int_0^t f(u)\, g(t-u)\, du .}$$

AKO KORISTIMO PRETHODNU JEDNAKOST, MOŽEMO PISATI DA JE

$$F(s) \cdot G(s) = \int_0^{+\infty} \left(f * g(t) \right) e^{-s \cdot t}\, dt = \mathcal{L}\{f * g\},$$

ODNOSNO, VAŽI TEOREMA O KONVOLUCIJI (BORELOVA TEOR): (P3)

$$F(s)\cdot G(s)=\mathcal{L}\{f*g\}$$

ILI

$$\boxed{\mathcal{L}\{f\}\cdot\mathcal{L}\{g\}=\mathcal{L}\{f*g\}}.$$

• OSOBINE KONVOLUCIJE

1) $f*g=g*f$

$$f*g=\int_0^t f(u)\,g(t-u)\,du=\int_t^0 f(t-v)\,g(v)(-dv)=$$

SMENA $u=t-v$ $\quad u=0\Rightarrow v=t$

$du=-dv$ $\quad u=t\rightarrow v=0$

$$=\int_0^t g(v)\,f(t-v)\,dv=g*f$$

2) $f*(cg)=(cf)*g=c(f*g)$, $c\in\mathbb{R}$

3) $f*(g+h)=f*g+f*h$

4) $f*(g*h)=(f*g)*h$

PRIMER IZRAČUNATI $f*g$, AKO JE $f(t)=t^2$, $g(t)=2t$.

REŠ.

$$f*g=t^2*(2t)=\int_0^t f(u)\cdot g(t-u)\,du=\int_0^t u^2\cdot 2(t-u)\,du=$$

$$=2\int_0^t (u^2 \cdot t - u^3)\,du = 2\cdot t\int_0^t u^2\,du - 2\int_0^t u^3\,du =$$

$$= 2t\cdot \frac{u^3}{3}\Big|_0^t - 2\frac{u^4}{4}\Big|_0^t = 2t\cdot\left(\frac{t^3}{3}-0\right) - 2\left(\frac{t^4}{4}-0\right) =$$

$$= \frac{2t^4}{3} - \frac{\cancel{2}t^4}{\cancel{4}} = \frac{2t^4}{3} - \frac{t^4}{2} = \frac{4t^4-3t^4}{6} = \frac{t^4}{6}$$

$$(f * g)(t) = \frac{t^4}{6}$$

II NAČIN (PRIMENOM TEOREME O KONVOLUCIJI)

$$\mathcal{L}\{t^2 * (2t)\} = \mathcal{L}\{t^2\}\cdot\mathcal{L}\{2t\} = \frac{2!}{s^3}\cdot 2\cdot\frac{1}{s^2} =$$

$$= 4\cdot\frac{1}{s^5} = \frac{4}{s^5}$$, DAKLE

$$t^2 * (2t) = \mathcal{L}^{-1}\{\mathcal{L}\{t^2 * (2t)\}\} = \mathcal{L}^{-1}\left\{\frac{4}{s^5}\right\} =$$

$$= 4\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{4}{4!}\,\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} =$$

$$= \frac{4}{4!}t^4 = \frac{\cancel{4}}{2\cdot 3\cdot \cancel{4}}t^4 = \frac{t^4}{6}$$

PRIMER REŠITI INTEGRO-DIFERENCIJALNU JEDNAČINU

$$y''(t) - y(t) = 6t - \int_0^t \sin(t-u)\,y(u)\,du\ ,\ \text{uz}$$

POČETNE USLOVE $y(0) = y'(0) = 0$.

REŠ: PRIMETIMO DA JE $\int_0^t \sin(t-u)\,y(u)\,du = y(t) * \sin(t)$

$$\mathcal{L}\left\{\int_0^t \sin(t-u)\,y(u)\,du\right\} = \mathcal{L}\{y(t) * \sin(t)\} = \mathcal{L}\{y(t)\}\cdot\mathcal{L}\{\sin(t)\}$$

$$=$$

$$= Y(s) \cdot \frac{1}{s^2+1}, \text{ DAKLE}$$

$$\underbrace{Y(s)}_{\mathcal{L}\{y\}} \cdot \underbrace{\frac{1}{s^2+1}}_{\mathcal{L}\{\sin t\}}$$

$$\mathcal{L}\left\{\int_0^t \sin(t-u)\, y(u)\, du\right\} = Y(s)\cdot\frac{1}{s^2+1}$$

PRIMENJUJUĆI LAPLASOVU TRANSFORMACIJU NA POČETNU JEDNAČINU, IMAMO

$$\mathcal{L}\{y''(t)\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{6t\} - \mathcal{L}\left\{\int_0^t \sin(t-u)\cdot y(u)\, du\right\}$$

$$s^2 Y(s) - Y(s) = \frac{6}{s^2} - \frac{1}{s^2+1}\cdot Y,$$

ODAKLE NALAZIMO DA JE

$$Y(s) = \frac{6s^2+6}{s^6} \Rightarrow Y(s) = \frac{6}{s^4} + \frac{6}{s^6}.$$

PRIMENOM INVERZNE LAP. TRANSFORMACIJE

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{s^6}\right\} =$$

$$= 6\cdot\frac{t^3}{3!} + 6\cdot\frac{t^5}{5!} = t^3 + \frac{t^5}{20}$$

$$y(t) = t^3 + \frac{t^5}{20}$$