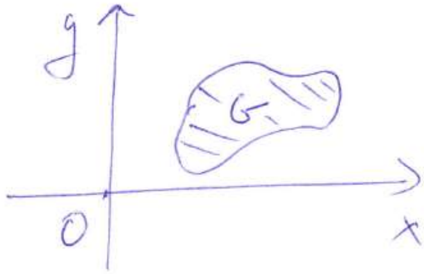
DVOSTRUKI INTEGRAL, IZRAČUNAVANJE I POLARNE KOORDINATE

(*) $$\iint_G f(x,y)\,dA = \iint_G f(x,y)\,dx\,dy$$



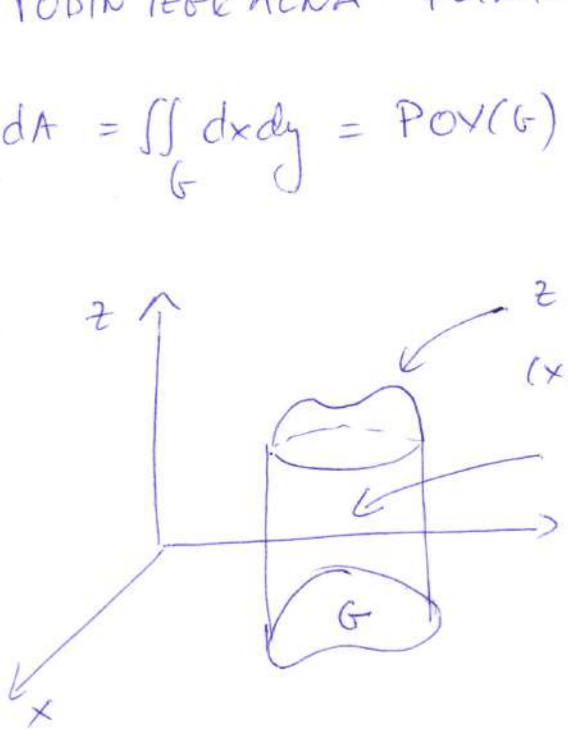
$G \subseteq \mathbb{R}^2$ (G JE OBLAST U xy-RAVNI)

G - DOMEN INTEGRACIJE

dA - ELEMENT POVRŠINE ILI ELEMENT DOMENA (AREA ELEMENT)

f - PODINTEGRALNA FUNKCIJA, ~~f~~ $f : G \to \mathbb{R}$

• $$\iint_G dA = \iint_G dx\,dy = POV(G)$$

• 

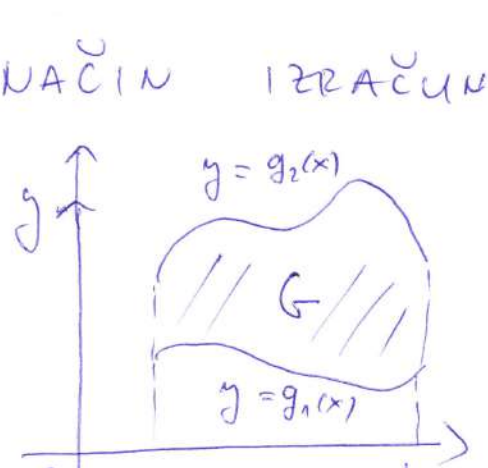
$z = f(x,y) > 0$

$(x,y) \in G$

$$ZAP(V) = \iint_G f(x,y)\,dx\,dy$$

V - CILINDRIČNA OBLAST U $\mathbb{R}^3$

(*) NAČIN IZRAČUNAVANJA

• 

~~G~~ GRANICE ZA G:

$a \le x \le b$

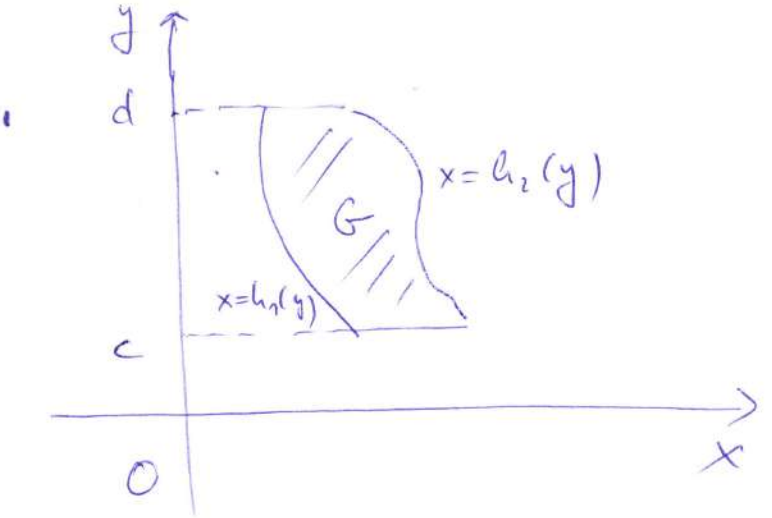
$g_1(x) \le y \le g_2(x)$

ILI

$$G = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

$$\iint_G f(x,y)\,dx\,dy = \int_a^b dx \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) = \dots$$

P2



GRANICE ZA G:

$c \le y \le d$

$h_1(y) \le x \le h_2(y)$

ILI

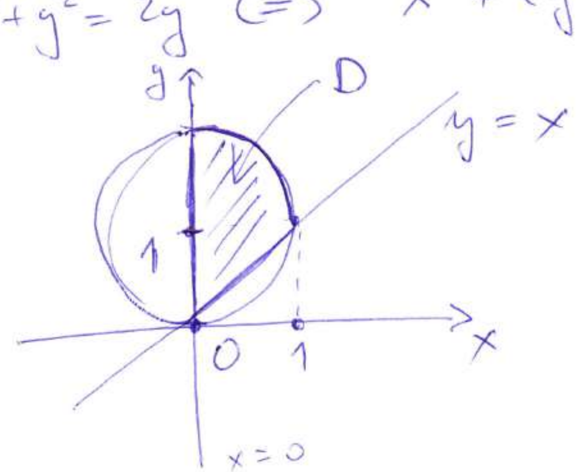
$G=\{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

$$\iint_G f(x,y)\,dx\,dy = \int_c^d dy \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx = \ldots$$

PRIMER IZRAČUNATI $\iint_D x\,dx\,dy$, AKO JE OBLAST D OGRANIČENA DELOM KRIVE $x^2+y^2=2y$ I PRAVAMA $y=x$ I $x=0$.

REŠ.

$x^2+y^2=2y \iff x^2+(y-1)^2=1$



ODREDIMO GRANICE OBLASTI G:

$0 \le x \le 1$

$x \le y \le 1+\sqrt{1-x^2}$

$x^2+(y-1)^2=1$

$(y-1)^2=1-x^2$

$y-1=\pm\sqrt{1-x^2}$

$\nearrow\ y=1+\sqrt{1-x^2}$ ← GORNJA POLUKRUŽNICA

$\searrow\ y=1-\sqrt{1-x^2}$

P3

$$\iint_D x\,dx\,dy = \int_0^1 dx \int_x^{1+\sqrt{1-x^2}} x\,dy = \int_0^1 dx \; xy\Big|_{x}^{1+\sqrt{1-x^2}} =$$

$$= \int_0^1 dx \cdot \left(x\left(1+\sqrt{1-x^2}\right) - x^2 \right) = \int_0^1 \left(x - x^2 + x\sqrt{1-x^2} \right) dx =$$

$$= \frac{x^2}{2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1 + \int_0^1 x\sqrt{1-x^2}\,dx = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} = \boxed{\frac{1}{2}}$$

SMENA:

$1 - x^2 = t$

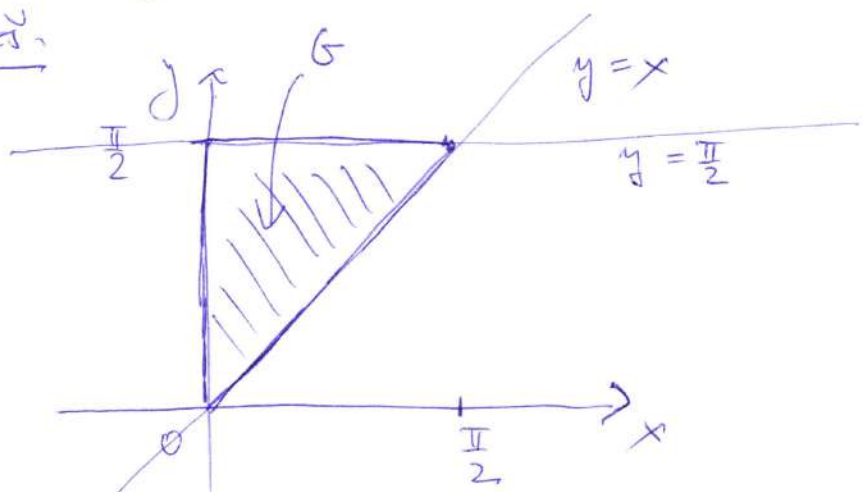
$-2x\,dx = dt \;\rightarrow\; x\,dx = -\frac{1}{2}dt$

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\int_1^0 \frac{1}{2}\sqrt{t}\,dt = -\frac{1}{2}\frac{t^{3/2}}{3/2}\Big|_1^0 = -\frac{1}{3}\,t\sqrt{t}\Big|_1^0 = \frac{1}{3}$$

PRIMER IZRAČUNATI $\iint_G \frac{\sin y}{y}\,dx\,dy$, AKO JE

$G = \{(x,y) \mid 0 < x < \frac{\pi}{2},\ x < y < \frac{\pi}{2}\}$.

REŠ.



$$\iint_G \frac{\sin y}{y}\,dx\,dy = \int_0^{\frac{\pi}{2}} dx \int_x^{\frac{\pi}{2}} \frac{\sin y}{y}\,dy = \text{MEĐUTIM, } \int \frac{\sin y}{y}\,dy = ?$$

PROMENOM REDOSLEDA INTEGRACIJE, OBLAST G MOŽEMO OPISATI I NA SLEDEĆI NAČIN

$$\boxed{G: \begin{array}{l} 0 \leq y < \frac{\pi}{2} \\ 0 < x < y \end{array}}$$

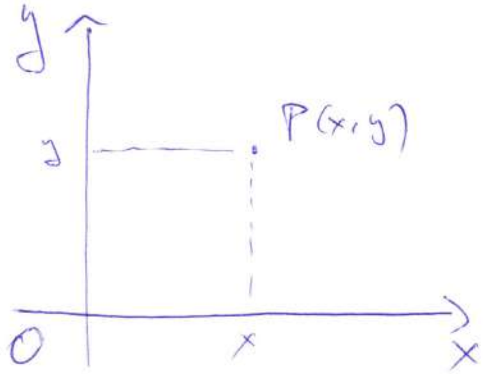
TADA, IMAMO

$$\iint_G \frac{\sin y}{y}\, dx\, dy = \int_0^{\frac{\pi}{2}} dy \int_0^{y} \frac{\sin y}{y}\, dx = \int_0^{\frac{\pi}{2}} dy \left(\frac{\sin y}{y} \cdot x \Big|_0^{y} \right) =$$

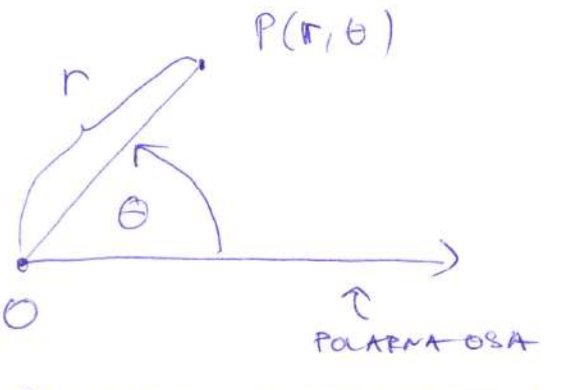
$$= \int_0^{\frac{\pi}{2}} dy\, \frac{\sin y}{y} \cdot (y - 0) = \int_0^{\frac{\pi}{2}} \frac{\sin y}{\cancel{y}} \cdot \cancel{y} \cdot dy = \int_0^{\frac{\pi}{2}} \sin y\, dy =$$

$$= -\cos y \Big|_0^{\frac{\pi}{2}} = -(\underbrace{\cos\tfrac{\pi}{2}}_{0} - \underbrace{\cos 0}_{1}) = \boxed{1}$$

POLARNE KOORDINATE



DEKARTOV ILI PRAVOUGLI KOORDINATNI SISTEM

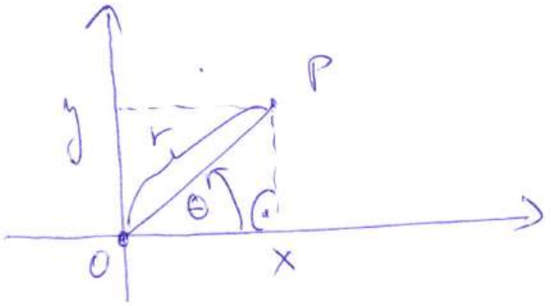


POLARNI KOORDINATNI SISTEM

$$P(x,y) \longleftrightarrow P(r,\theta) \quad r \geq 0,\ \theta \in [0, 2\pi]$$

r I θ SU POLARNE KOORDINATE

VEZA IZMEĐU PRAVOUGLIH I POLARNIH KOORDINATA: P5

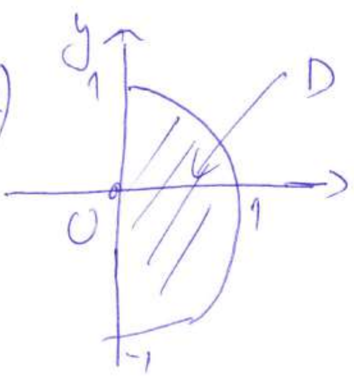


$$\sin\theta = \frac{y}{r} \Rightarrow \boxed{y = r\sin\theta}$$

$$\cos\theta = \frac{x}{r} \Rightarrow \boxed{x = r\cos\theta}$$

PRIMER. ODREDITI GRANICE ~~ZA~~ OBLASTI D U ODNOSU NA POLARNE KOORDINATE, AKO JE

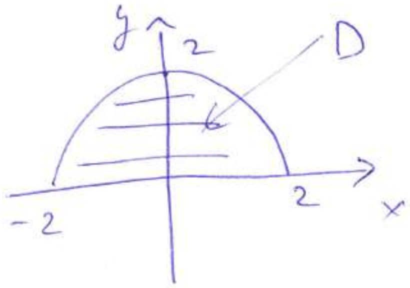
a)



$D: \quad 0 \le r \le 1$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

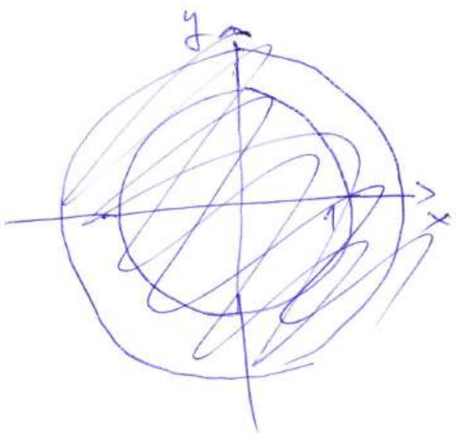
b)

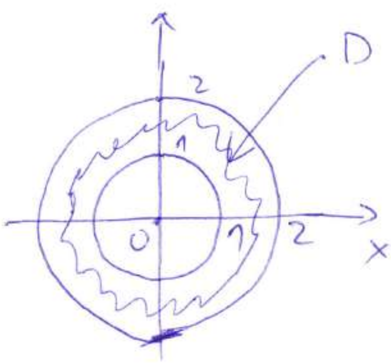


$D: \quad 0 \le r \le 2$

$0 \le \theta \le \pi$

c)



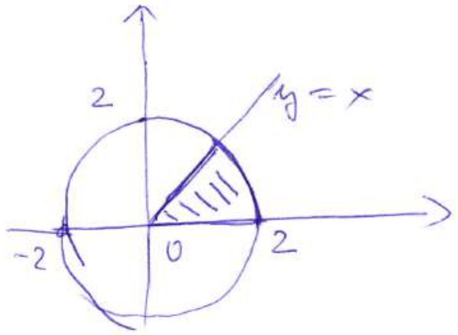


D (PRSTEN)

$D: \quad 1 \le r \le 2$

$0 \le \theta \le 2\pi$

d)



$D: \quad 0 \le r \le 2$

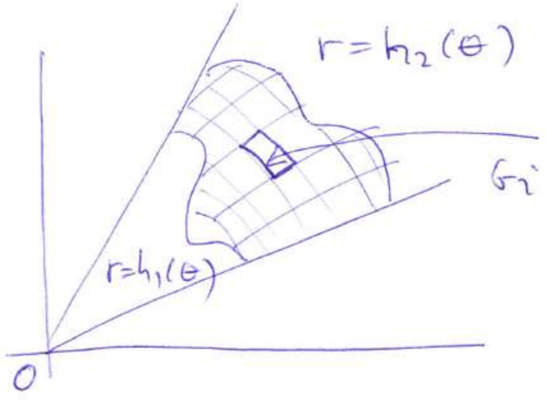
$0 \le \theta \le \frac{\pi}{4}$

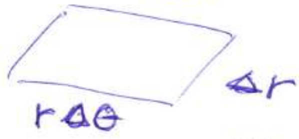
GRANICE SU KONSTANTNE

P6

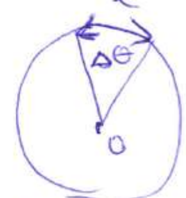
(*) ELEMENT POVRŠINE dA U ODNOSU NA POLARNE KOORDINATE

$$\iint_G f(x,y)\,dA \quad , \quad dA = dx\,dy$$, AKO JE G U PRAVOUGLOM KOORDINATNOM SISTEMU





KRUG



$l = \Delta\theta \cdot r$

ΔG_i JE POV. KRIVOLINIJSKOG ČETVOROUGLA $\approx$ PRAVOUGAONIK ZA MALE VREDNOSTI Δr, $\Delta\theta$

$r \to r + \Delta r$

$\theta \to \theta + \Delta\theta$

$$\Delta G_i = r \cdot \Delta\theta \cdot \Delta r$$

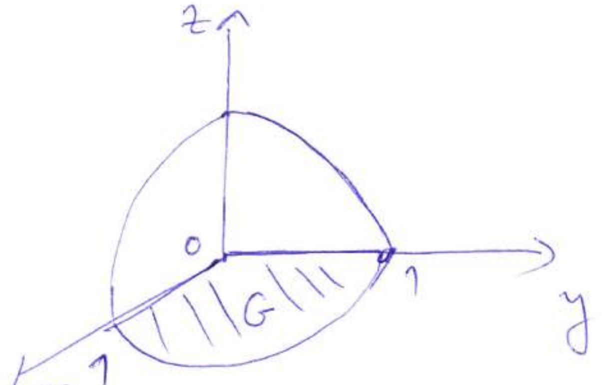
$$\Downarrow$$

$$dA = r \cdot d\theta \cdot dr$$

DAKLE, VAŽI SLEDEĆA VEZA PRELAZA:

$$\iint_G f(x,y)\,\underset{dx\,dy}{dA} = \iint_{G^*} f(r\cos\theta, r\sin\theta)\cdot r \cdot d\theta\, dr$$

PRIMER. ODREDITI ZAPREMINU OBLASTI ISPOD PARABOLOIDA $z = 1 - x^2 - y^2$ U PRVOM OKTANTU.



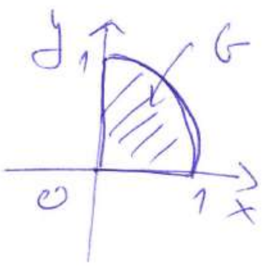
$z = 0 \Rightarrow x^2 + y^2 = 1$

$x = r\cos\theta$

$y = r\sin\theta$

$G: \; 0 \le r \le 1$

$\quad\; 0 \le \theta \le \frac{\pi}{2}$



P7

$\cancel{V: (x,y) \in G}$

$$ZAP = \iint_G (1 - x^2 - y^2 - 0)\,dx\,dy = \iint_G (1 - x^2 - y^2)\,dx\,dy =$$

$$= \int_0^1 dr \int_0^{\frac{\pi}{2}} \left(1 - (r\cos\theta)^2 - (r\sin\theta)^2\right) r\,d\theta =$$

$$= \int_0^1 dr \cdot r \cdot \int_0^{\frac{\pi}{2}} \left(1 - r^2\cos^2\theta - r^2\sin^2\theta\right) d\theta =$$

$$= \int_0^1 dr \cdot r \cdot \int_0^{\frac{\pi}{2}} \left(1 - r^2 (\underbrace{\cos^2\theta + \sin^2\theta}_{=1})\right) d\theta = \int_0^1 r \cdot dr \int_0^{\frac{\pi}{2}} (1 - r^2)\,d\theta =$$

$$= \int_0^1 r(1 - r^2)\,dr \cdot \theta \Big|_0^{\frac{\pi}{2}} = \int_0^1 r(1 - r^2)\,dr \cdot \left(\frac{\pi}{2} - 0\right) =$$

$$= \frac{\pi}{2} \int_0^1 (r - r^3)\,dr = \frac{\pi}{2}\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1 = \frac{\pi}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{2} \cdot \frac{1}{4} =$$

$$= \boxed{\frac{\pi}{8}}$$