UVOĐENJE SMENE KOD DVOSTRUKOG INTEGRALA

(*) UVODNI POJMOVI

U ANALIZI 1 ZA REŠAVANJE ODREĐENOG INTEGRALA KORISTI SE METODA SMENE.

$$\int_a^b f(x)\,dx = \int_{t_1}^{t_2} f(\varphi(t))\cdot \varphi'(t)\,dt$$

SMENA: $x = \varphi(t)$ GDE JE

$dx = \varphi'(t)\,dt$ $\quad t_1 = \varphi^{-1}(a)$, $\quad t_2 = \varphi^{-1}(b)$

U ANALIZI 2 SLEDIMO ISTU IDEJU.

$$\iint_G f(x,y)\,dx\,dy$$

UVODIMO SMENU: $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ $\xleftrightarrow{\text{PRETPOSTAVLJAMO DA JE MOGUĆE}}$ $\begin{array}{l} u = u(x,y) \\ v = v(x,y) \end{array}$

- (x,y) STARE PROM. $\longrightarrow$ (u,v) NOVE PROM.

  G OBLAST INT. U ODNOSU NA x i y $\qquad$ G^* OBLAST INT. U ODNOSU NA ~~FJ~~ u i v

- $f(x,y) \longrightarrow f(x(u,v), y(u,v))$

- $dA = dx\,dy \longrightarrow dA = ?$

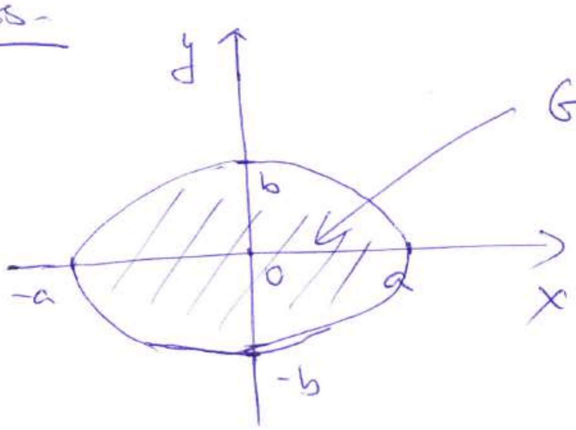
  TREBA ODREDITI ELEMENT POVRŠINE

(P2)

PRIMER IZRAČUNATI $\iint_G dx\,dy$, AKO JE

$$G = \left\{(x,y) \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1\right\}.$$

REŠ.



G, G JE ~~EL~~ ELIPSA SA POLUOSAMA a i b

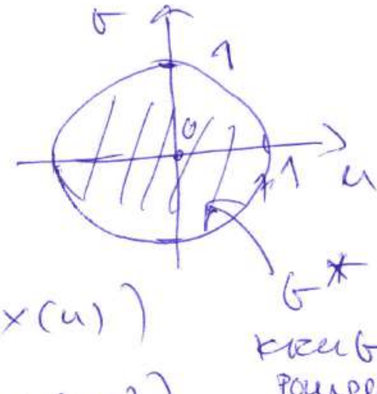
$\iint_G dx\,dy = POV(G)$, POV. ELIPSE ~~POV ...~~

$$\iint_G dx\,dy = ?$$

UVEDIMO SMENU: $\boxed{\frac{x}{a} = u \ , \ \frac{y}{b} = v}$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1 \;\rightarrow\; u^2 + v^2 \leq 1 \text{, DAKLE}$$

$$G^* = \{(u,v) \mid u^2 + v^2 \leq 1\} \rightarrow$$



G^* KRUG POLUPREČ. 1

$\frac{x}{a} = u \rightarrow x = au \rightarrow dx = a\,du \quad (x = x(u))$

$\frac{y}{b} = v \rightarrow y = bv \rightarrow dy = b\,dv \quad (y = y(v))$

$dx\,dy = ab \cdot du\,dv \qquad (dA = ab\,du\,dv)$

$$\boxed{\iint_G dx\,dy = \iint_{G^*} ab\,du\,dv = ab \underbrace{\iint_{G^*} du\,dv}_{r^2\pi = \pi} = ab \cdot \pi}$$

* UVOĐENJE OPŠTE SMENE KOD DVOSTRUKOG INTEGRALA

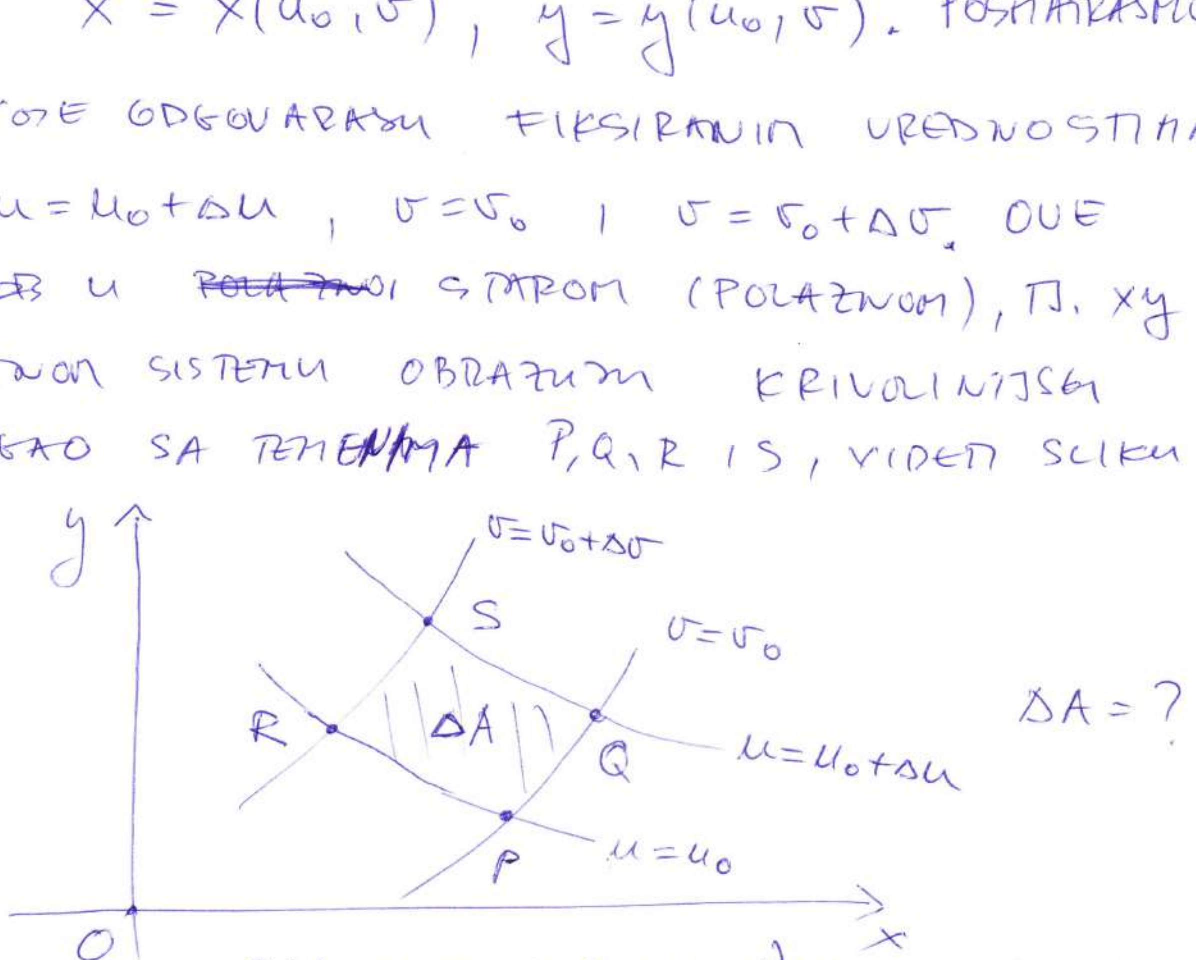
POSMATRAMO VEZU (SMENU) IZMEĐU STARIH I NOVIH PROMENLJIVIH:

$$x = x(u,v), \quad y = y(u,v)$$

PRETPOSTAVIMO DA JE MOGUĆE IZRAZITI I

$$u = u(x,y), \quad v = v(x,y).$$

POTREBNO JE ODREDITI $dA = dx\,dy$ PRI PRELASKU SA ~~xy~~ x i y NA u i v. FIKSIRANJEM $u = u_0$ (ILI $v = v_0$) U STAROM KOORDINATNOM SISTEMU DEFINIŠEMO KRIVU: $x = x(u_0, v)$, $y = y(u_0, v)$. POSMATRAMO KRIVE KOJE ODGOVARAJU FIKSIRANIM VREDNOSTIMA $u = u_0$, $u = u_0 + \Delta u$, $v = v_0$ I $v = v_0 + \Delta v$. OVE KRIVE ~~OB~~ U ~~POLAZNOJ~~ STAROM (POLAZNOM), TJ. xy KOORDINATNOM SISTEMU OBRAZUJU KRIVOLINIJSKI ČETVOROUGAO SA TEMENIMA P, Q, R I S, VIDETI SLIKU.



$S(x(u_0+\Delta u, v_0+\Delta v), y(u_0+\Delta u, v_0+\Delta v))$ ~~$S(x(u_0+\Delta u, v_0+\Delta v), y(u_0+\Delta u, v_0+\Delta v))$~~

$P(x(u_0,v_0), y(u_0,v_0))$, $Q(x(u_0+\Delta u, v_0), y(u_0+\Delta u, v_0))$, $R(x(u_0, v_0+\Delta v), y(u_0, v_0+\Delta v))$

TADA VAŽI,

$$\vec{PQ} = \left(x(u_0+\Delta u, v_0), y(u_0+\Delta u, v_0)\right) - \left(x(u_0, v_0), y(u_0, v_0)\right)$$

$$= \left(x(u_0+\Delta u, v_0) - x(u_0, v_0), \; y(u_0+\Delta u, v_0) - y(u_0, v_0)\right),$$

MEĐUTIM, $x(u_0+\Delta u, v_0) - x(u_0, v_0) = \underbrace{\dfrac{x(u_0+\Delta u, v_0) - x(u_0, v_0)}{\Delta u}}_{\approx \frac{\partial x}{\partial u} = x'_u} \cdot \Delta u \approx$

$\approx x'_u \cdot \Delta u \quad (u = u_0)$, SLIČNO

$$y(u_0+\Delta u, v_0) - y(u_0, v_0) \approx y'_u \cdot \Delta u,$$

DAKLE, DOBILI SMO SLEDEĆU APROKSIMACIJU DUŽI $\vec{PQ}$:

$\vec{PQ} \approx \left(x'_u \cdot \Delta u, \; y'_u \cdot \Delta u\right)$, NA SLIČAN NAČIN

DOBIJAMO I DA JE $\vec{PR} \approx \left(x'_v \cdot \Delta v, \; y'_v \Delta v\right)$.

KRIVOLINIJSKI ČETVOROUGAO P, Q, R, S, ZA MALE VREDNOSTI Δu I Δv, SE MOŽE APROKSIMIRATI PARALELOGRAMOM. STRANICE TOG PARALELOGRAMA SU ODREĐENE VEKTORIMA $\vec{PQ}$ I $\vec{PR}$.

ZA POVRŠINU OVOG PARALELOGRAMA MOŽEMO PISATI

$$\Delta A = \underbrace{|\vec{PQ} \times \vec{PR}|}_{\text{VEKT. NORMA}} \approx \left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u \Delta u & y_u \Delta u & 0 \\ x'_v \cdot \Delta v & y'_v \Delta v & 0 \end{vmatrix} \right| =$$

($z = 0$, JER SU $\vec{PQ}$ I $\vec{PR}$ U RAVNI)

$$= \left| \begin{vmatrix} x'_u \Delta u & y'_u \Delta u \\ x'_v \Delta v & y'_v \Delta v \end{vmatrix} \vec{k} \right| = \left| \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix} \right| \cdot \Delta u \cdot \Delta v$$

$\vec{k} = (0, 0, 1)$ ← APS. VREDNOST

P5

$$\Delta A \approx \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \right| \Delta u \cdot \Delta v$$

APS. VREDNOST

U GRANIČNO SLUČAJU, KADA $\Delta u \to 0$, $\Delta v \to 0$, IMAMO:

$$dA = dx\,dy = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \right| \cdot du\,dv.$$

DETERMINANTU $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ NAZIVAMO

<u>JAKOBIJAN</u> I OBELEŽAVAMO SA J, ČESTO PIŠEMO

$D = D^T$

$$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

JAKOBIJEVA MATRICA

DAKLE, OPŠTA FORMULA ZA UVOĐENJE SMENE U DVOSTRUKI INTEGRAL

$$\boxed{\iint_G f(x,y)\,dx\,dy = \iint_{G^*} f(x(u,v), y(u,v))\,|J|\,du\,dv.}$$

- VAŽI SLEDEĆA VEZA: $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| \cdot \left| \frac{\partial(u,v)}{\partial(x,y)} \right| = 1 \Leftrightarrow$

$$\Leftrightarrow \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{\left| \frac{\partial(u,v)}{\partial(x,y)} \right|}$$

(P6)

PRIMER POKAZATI DA JE JAKOBIJAN PRI UVOĐENJU POLARNIH KOORDINATA JEDNAKO r.

Reš.

$$\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \end{cases}$$

"$x(u,v) = r\cos\theta$"

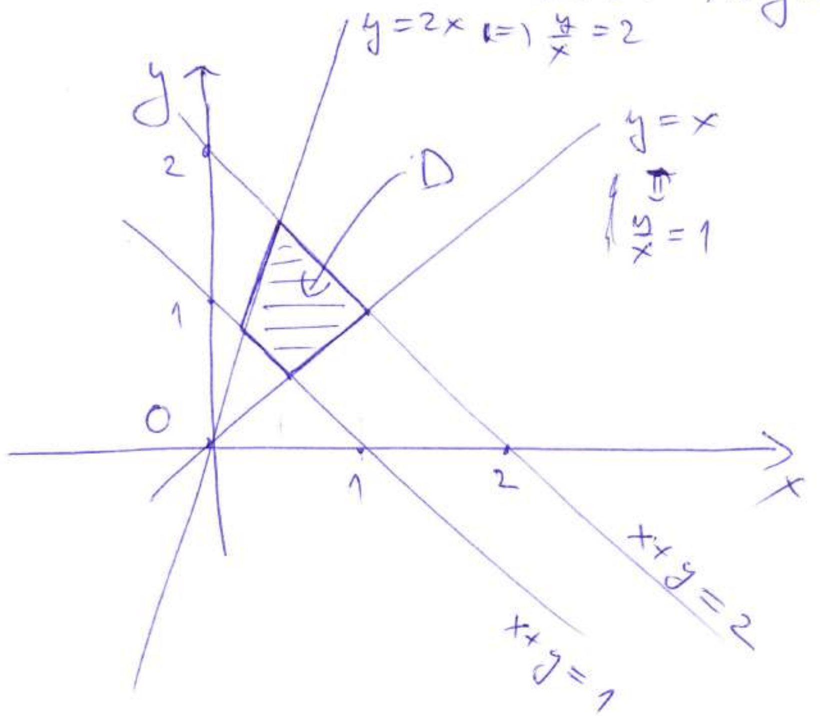
"$y(u,v) = r\sin\theta$" $\rightarrow$ $u, v \leftrightarrow r, \theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r \cdot \sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta =$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

PRIMER IZRAČUNATI $\iint_D dx\,dy$, AKO JE OBLAST D DEFINISANA NEJEDNAČINAMA: $x+y \geq 1$, $x+y \leq 2$, $y \geq x$ i $y \leq 2x$.

Reš.



a) DIREKTNO (BEZ SMENE)

$$D = D_1 \cup D_2 \cup D_3$$

$$\iint_D = \iint_{D_1} + \iint_{D_2} + \iint_{D_3} = \ldots$$

KOMPLIKOVANO...

b) POSMATRAJMO SMENU:

$$\left.\begin{aligned} x+y &= u \\ \frac{y}{x} &= v \end{aligned}\right. \longrightarrow D^*: \begin{cases} 1 \leq u \leq 2 \\ 1 \leq v \leq 2 \end{cases}$$

$$\iint_D dx\,dy = \iint_{D^*} |J|\,du\,dv$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{1}{x} + \frac{y}{x^2} = \frac{x+y}{x^2} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|$$

$$J = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{\frac{x+y}{x^2}} = \frac{x^2}{x+y} = \frac{\frac{u^2}{(1+v)^2}}{u} = \frac{u}{(1+v)^2}$$

$$x+y=u \rightarrow x = u-y \rightarrow x = u - xv \rightarrow x = \frac{u}{1+v}$$

$$\frac{y}{x} = v \rightarrow y = x\cdot v$$

$$\iint_{D^*} |J|\,du\,dv = \int_1^2 du \int_1^2 \frac{u}{(1+v)^2}\cdot dv = \int_1^2 u\,du \int_1^2 \frac{dv}{(1+v)^2} =$$

$$1+v = t$$
$$dv = dt$$

$$= \int_1^2 u\,du \left(-\frac{1}{1+v}\Big|_1^2 \right) = \frac{1}{6}\int_1^2 u\,du = \frac{1}{2}\,\frac{u^2}{2}\Big|_1^2 = \boxed{\frac{1}{4}}$$

* PRIMENA DVOSTRUKOG INTEGRALA

1) IZRAČUNAVANJE ZAPREMINE OBLASTI U 3D

$z = f(x,y)$

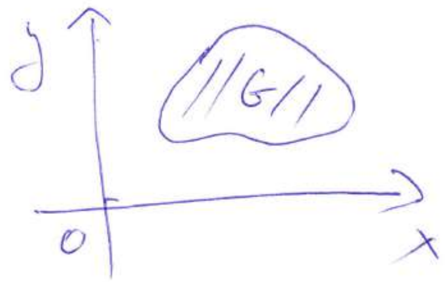
$z = g(x,y)$

$$\text{ZAP} = \iint_G \big(f(x,y) - g(x,y)\big)\,dx\,dy$$

G $\quad f(x,y) \geq g(x,y)$

PF

2) IZRAČUNAVANJE POVRŠINE U 2D



$$POV(G) = \iint_G dx\,dy$$

3) IZRAČUNAVANJE SREDNJE VREDNOSTI FUNKCIJE

$$\cancel{\frac{\iint_G f(x,y)\,dx\,dy}{\iint_G f(x,y)\,dx}} \qquad \frac{\iint_G f(x,y)\,dx\,dy}{\iint_G dx\,dy} = f_{SR}$$

f_{SR} JE SREDNJA VREDNOST FUNKCIJE $f(x,y)$

NAD OBLAŠĆU $G \subseteq \mathbb{R}^2$.

4) IZRAČUNAVANJE TEŽIŠTA RAVNE FIGURE

$T(\bar{x}, \bar{y}) \Leftarrow$ TEŽIŠTE FIGURE G

$\cancel{\bar{x} = \iint_G x}$

$$\cancel{\bar{x} = \iint} \qquad \bar{x} = \frac{\iint_G x\,dx\,dy}{\iint_G dx\,dy}, \quad \bar{y} = \frac{\iint_G y\,dx\,dy}{\iint_G dx\,dy}$$

"SREDNJA VREDNOST x i y KOORDINATA U G."