# TROSTRUKI INTEGRAL



(*) DEFINICIJA TROSTRUKOG INTEGRALA

NEKA JE $f(x,y,z)$ FUNKCIJA DEFINISANA NAD $D \subseteq \mathbb{R}^3$, $f: D \to \mathbb{R}$. NEKA JE $f$ NEPREKIDNA NAD $D$.

POSMATRAJMO PODELU OBLASTI $D$ NA $n$ DISJUNKTNIH PODOBLASTI; $D = D_1 \cup D_2 \cup D_3 \cup \dots \cup D_n$.

~~OZNA~~ OZNAČIMO SA $\Delta D_i$ $(i = 1,2,\dots,n)$ ZAPREMINU (MERU) PODOBLASTI $D_i$, A SA $(x^i, y^i, z^i)$ PROIZVOLJNU TAČKU IZ $D_i$.

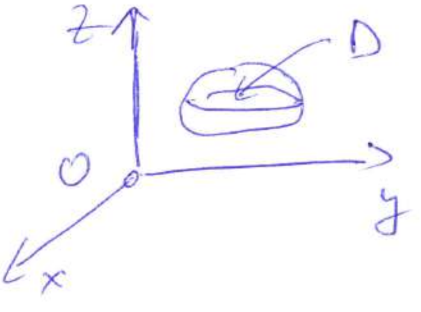
RIMANOVA INTEGRALNA SUMA FUNKCIJE $f$ NAD $D$:

$$\sum_{i=1}^{n} f(x^i, y^i, z^i) \cdot \Delta D_i .$$

TROSTRUKI INTEGRAL FUNKCIJE $f$ NAD $D$ JE GRANIČNA VREDNOST NIZA RIMANOVIH INT. SUMA:

$$\iiint\limits_D f(x,y,z)\, dV \overset{\text{df.}}{=\!=} \lim_{\Delta D_i \to 0} \sum_{i=1}^{n} f(x^i, y^i, z^i) \Delta D_i$$

$dV$ - ELEMENT ZAPREMINE $(dV \approx \Delta D_i)$




$D \subseteq \mathbb{R}^3$

## (*) OSOBINE TROSTRUKOG INTEGRALA

1) $\iiint\limits_D \alpha f(x,y,z)\,dV = \alpha \iiint\limits_D f(x,y,z)\,dV, \quad \alpha \in \mathbb{R}$

2) $\iiint\limits_D \big(f(x,y,z) \pm g(x,y,z)\big)\,dV = \iiint\limits_D f(x,y,z)\,dV \pm \iiint\limits_D g(x,y,z)\,dV$

3) $\iiint\limits_D f(x,y,z)\,dV = \iiint\limits_{D_1} f(x,y,z)\,dV + \iiint\limits_{D_2} f(x,y,z)\,dV$, GDE JE

$D = D_1 \cup D_2$ I $D_1 \cap D_2 = \emptyset$

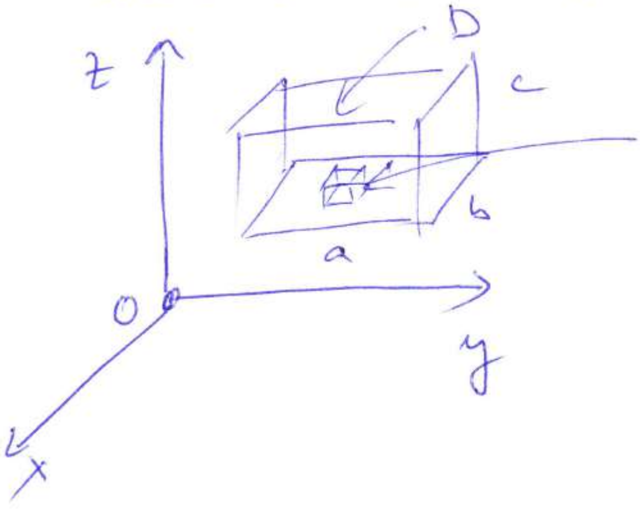
4) $\iiint\limits_D dV = ZAP(D)$

• AKO JE ~~f~~ $\rho(x,y,z)$ GUSTINA TELA $D \subseteq \mathbb{R}^3$ U TAČKI $(x,y,z) \in D$, ONDA JE $\iiint\limits_D \rho(x,y,z)\,dV$ MASA TELA D.

## (*) IZRAČUNAVANJE TROSTRUKOG INTEGRALA

a) ~~OI~~ OBLAST INTEGRACIJE D JE KVADAR U $\mathbb{R}^3$, ODNOSNO

$D = [a,b] \times [c,d] \times [g,h]$




$D_i \to \Delta D_i = \Delta x_i \cdot \Delta y_i \cdot \Delta z_i$

$dV \approx \Delta D_i = \Delta x_i \cdot \Delta y_i \cdot \Delta z_i$

$dV = dx\,dy\,dz$

P3

D: $a \le x \le b$
$c \le y \le d$ TADA,
$g \le z \le h$

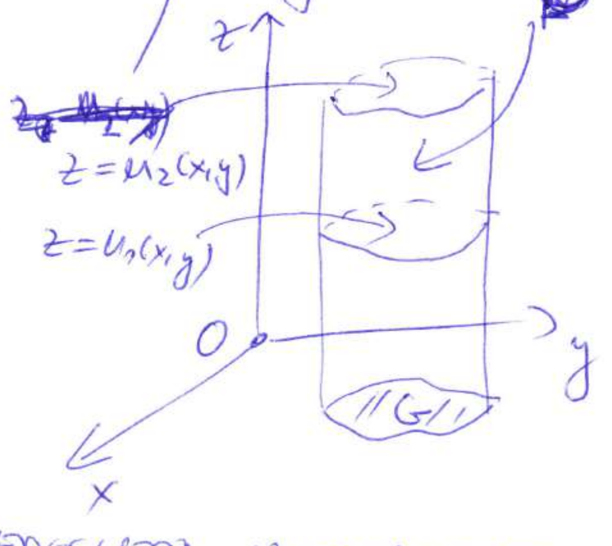
$$\iiint_D f(x,y,z)\,dV = \iiint_D f(x,y,z)\,dx\,dy\,dz =$$
$$= \int_a^b dx \left[ \int_c^d dy \left( \int_g^h f(x,y,z)\,dz \right) \right] = \int_a^b dx \int_c^d dy \int_g^h f(x,y,z)\,dz$$

b) OBLAST INT. $D = \{(x,y,z) \mid (x,y) \in G,\ u_1(x,y) \le z \le u_2(x,y)\}$,

A GDE JE $G = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$.

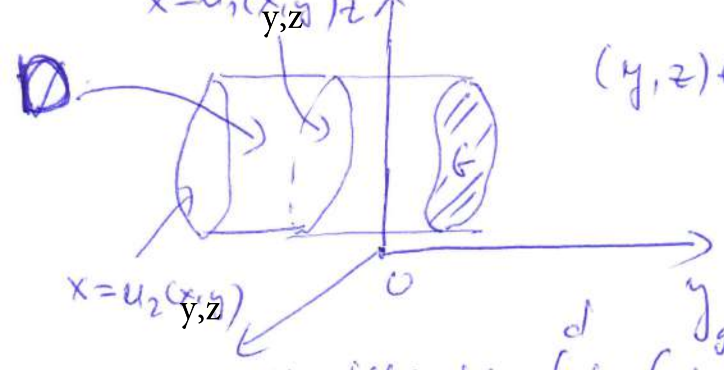
TADA JE

$$\iiint_D f(x,y,z)\,dV = \iint_G \left( \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dx\,dy =$$
$$= \int_a^b dx \int_{g_1(x)}^{g_2(x)} dy \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz$$

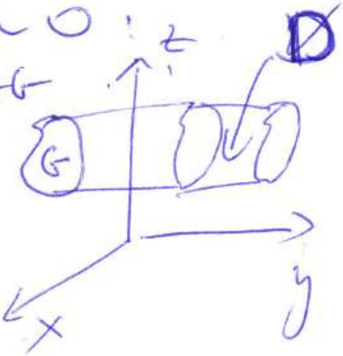



MEĐUTIM, MOGUĆA JE PROMENA REDOSLEDA INTEGRACIJE

$(y,z) \in G$




SLIČNO: $(x,z) \in G$

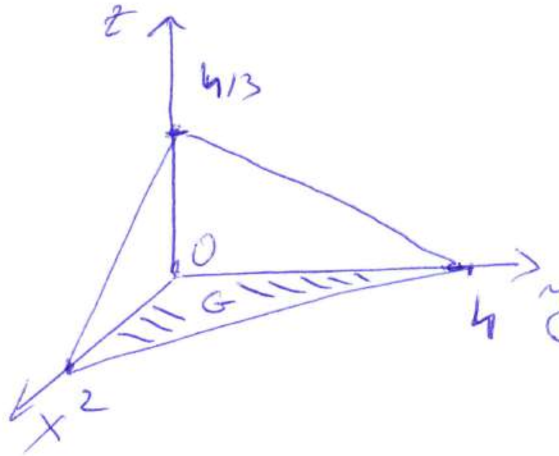



$$\iiint_D f\,dV = \int_c^d dz \int_{g_1(z)}^{g_2(z)} dy \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx$$

PRIMER IZRAČUNATI $\iiint\limits_V x\,dx\,dy\,dz$, AKO JE (R4)

V OBLAST OGRANIČENA ~~SA RAVNIMA~~ RAVNIMA $2x+y+3z=4$, $x=0$, $y=0$ i $z=0$.

Reš.




G JE PROJEKCIJA V NA xOy RAVAN

$$2x+y+3z=4$$
$$\Updownarrow$$
$$z=\frac{1}{3}(4-2x-y)$$

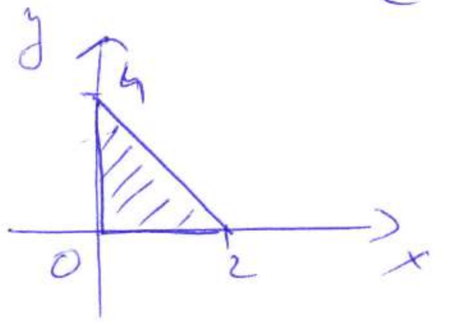
$x=0 \wedge y=0 \Rightarrow 2x+y+3z=4 \Leftrightarrow z=4/3$

$x=0 \wedge z=0 \Rightarrow 2x+y+3z=4 \Leftrightarrow y=4$

$y=0 \wedge z=0 \Rightarrow 2x+y+3z=4 \Leftrightarrow x=2$

GRANICE OBLASTI V: $\begin{cases} (x,y)\in G \\ 0\le z\le \frac{1}{3}(4-2x-y) \end{cases}$

V IZ G:




$0\le x\le 2$
$0\le y\le 4-2x$, DAKLE

$$V:\begin{cases} 0\le x\le 2 \\ 0\le y\le 4-2x \\ 0\le z\le \frac{1}{3}(4-2x-y) \end{cases}, \quad \iiint\limits_V x\,dx\,dy\,dz =$$

$$= \int_0^2 dx \int_0^{4-2x} dy \int_0^{\frac{1}{3}(4-2x-y)} x\,dz = \int_0^2 x\,dx \int_0^{4-2x} dy \cdot z\Big|_0^{\frac{1}{3}(4-2x-y)} =$$

$$= \int_0^2 x\,dx \int_0^{4-2x} \frac{1}{3}(4-2x-y)\,dy = \frac{1}{3}\int_0^2 x\,dx \int_0^{4-2x} (4-2x-y)\,dy \overset{\text{PS}}{=}$$

$$= \frac{1}{3}\int_0^2 x\,dx \left[4y - 2xy - \frac{y^2}{2}\right]\Bigg|_0^{4-2x} = \frac{1}{3}\int_0^2 x\,dx \left[4(4-2x) - 2x(4-2x) - \frac{(4-2x)^2}{2} - 0\right]$$

$$= \frac{1}{3}\int_0^2 \left(4x(4-2x) - 2x^2(4-2x) - \frac{1}{2}x\cdot(4-2x)^2\right)dx = \ldots = \boxed{\frac{8}{9}}$$

* UVOĐENJE SMENE KOD TROSTRUKOG INTEGRALA

POSMATRAJMO SMENU: $x = x(u,v,w)$, $y = y(u,v,w)$
$z = z(u,v,w)$

$$(x,y,z) \longrightarrow (u,v,w)$$
$$D \qquad\qquad D^*$$

PRETPOSTAVIMO DA JE PRESLIKAVANJE BIJEKTIVNO!

SLIČNO KAO KOD DVOSTRUKOG INTEGRALA MOŽEMO DOĆI DO FORMULE ZA UVOĐENJE SMENE:

$$\boxed{\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \cdot |J| \cdot du\,dv\,dw,}$$

GDE JE $J$ <u>JAKOBIJAN TRANSFORMACIJE (SMENE)</u>

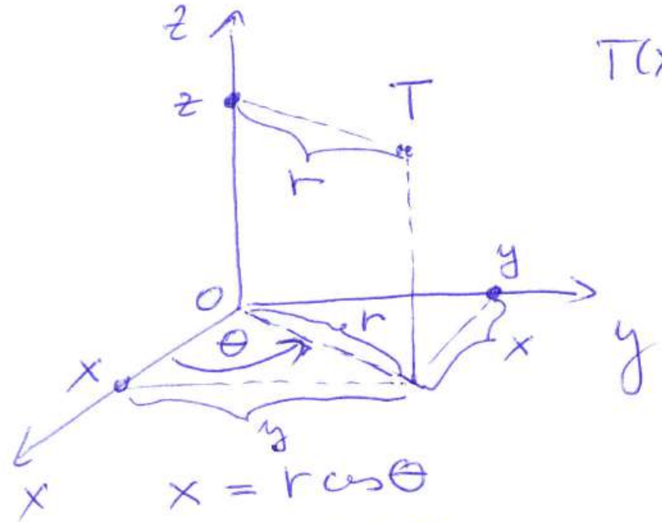
$$J = \begin{vmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{vmatrix} = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$$

P6

AKO JE $u = u(x,y,z)$, $v = v(x,y,z)$, $w = w(x,y,z)$

TADA VAŽI:
$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|}$$

## CILINDRIČNE KOORDINATE




$T(x,y,z) \Leftrightarrow T'(r,\theta,z)$

CILINDRIČNE KOORDINATE

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

VEZA IZMEĐU PRAVOUGLIH I CILINDRIČNIH KOORDINATA:

$$\boxed{\begin{aligned} x &= r\cdot\cos\theta & r &\geq 0 \\ y &= r\cdot\sin\theta, & 0 &\leq \theta \leq 2\pi \\ z &= z & z &\in \mathbb{R} \end{aligned}}$$

ZA JAKOBIJAN VAŽI:

$$J = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \begin{vmatrix} x'_r & x'_\theta & x'_z \\ y'_r & y'_\theta & y'_z \\ z'_r & z'_\theta & z'_z \end{vmatrix} =$$

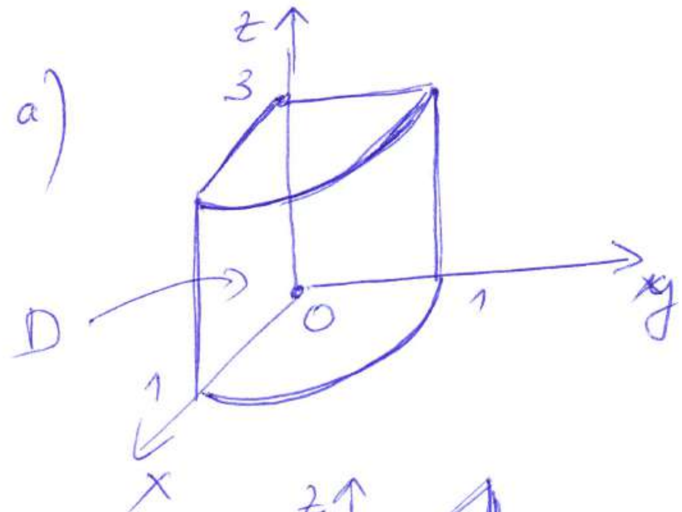
$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\boxed{J = r}$$

- CILINDRIČNE KOORDINATE SU POGODNE ZA OPISIVANJE OBLASTI KOJE IMAJU OSU SIMETRIJE PARALELNU NEKOJ OD KOORDINATNIH OSA! (CILINDRIČNA TELA)

PRIMER ODREDITI GRANICE SLEDEĆIH OBLASTI: (P7)

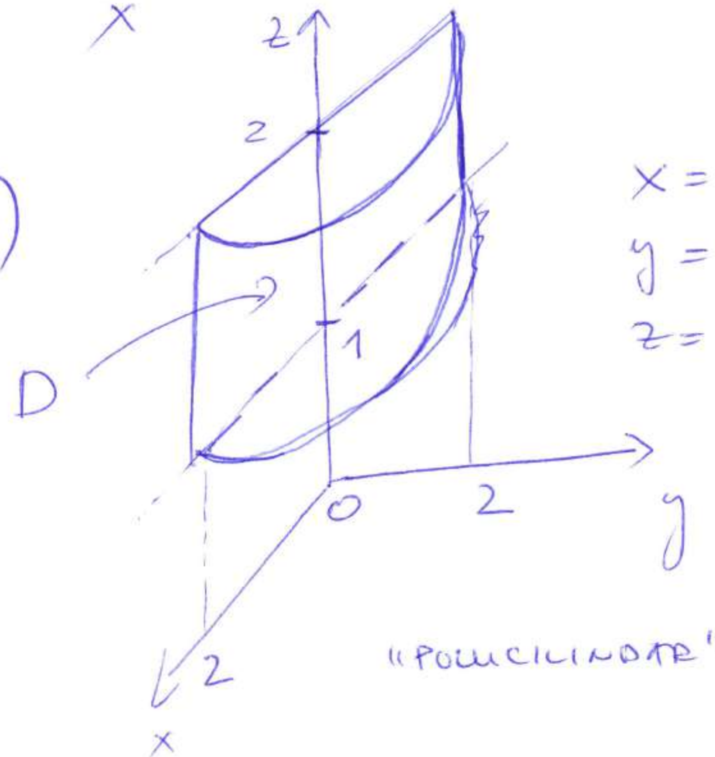
a)




$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$\Rightarrow D^*: \begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le \frac{\pi}{2} \\ 0 \le z \le 3 \end{cases}$

b)

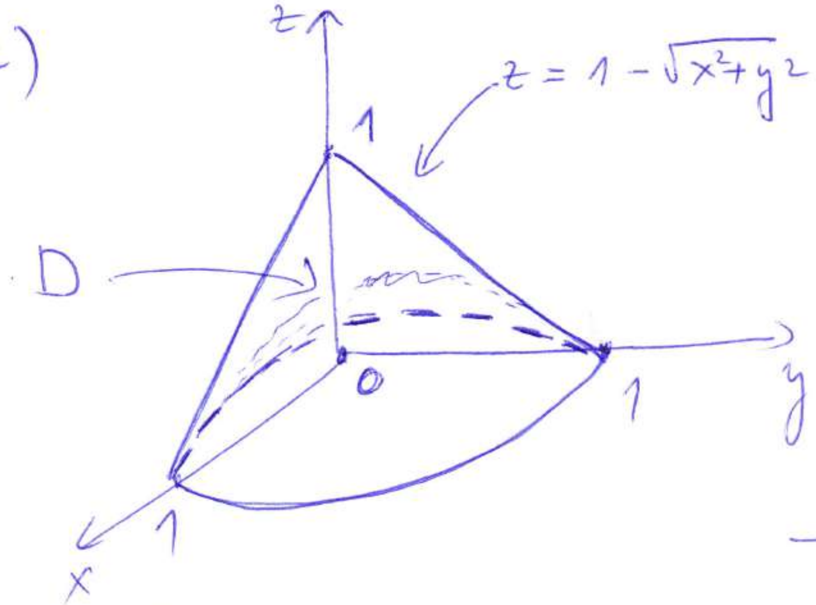



"POLUCILINDAR"

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$\Rightarrow D^*: \begin{cases} 0 \le r \le 2 \\ 0 \le \theta \le \pi \\ 1 \le z \le 2 \end{cases}$

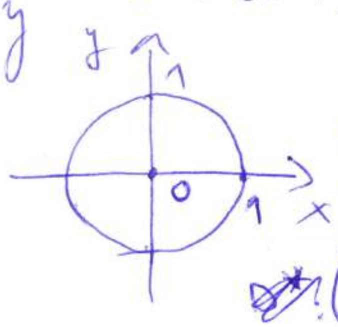
c)




$D = \{(x,y,z) \mid 0 \le z \le 1 - \sqrt{x^2+y^2}\}$

$z = 0 \Rightarrow 1 - \sqrt{x^2+y^2} = 0$

$\Leftrightarrow \sqrt{x^2+y^2} = 1 \Leftrightarrow x^2+y^2 = 1$



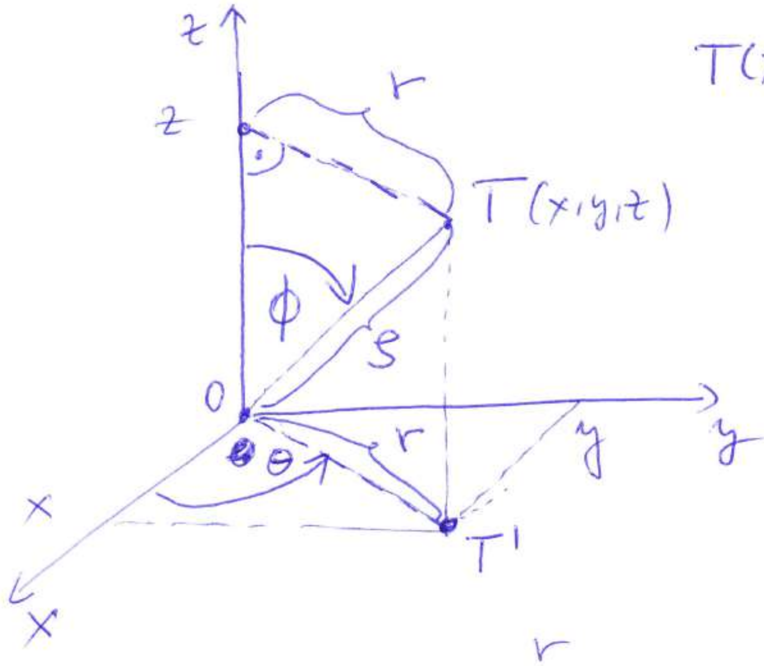

$D^*: \begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \\ 0 \le z \le 1 - r \end{cases}$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$z = 1 - \sqrt{x^2+y^2}$
$\Downarrow$
$z = 1 - \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = 1 - r$

# (*) SFERNE KOORDINATE

(PF)




$T(x,y,z) \longleftrightarrow$ [redacted] $T^*(\rho,\theta,\phi)$

$\Delta ZOT$:

$$\cos\phi = \frac{z}{\rho} \Rightarrow z = \rho\cos\phi$$

$$\sin\phi = \frac{r}{\rho} \Rightarrow r = \rho\sin\phi,$$

KORISTEĆI POLARNE KOORDINATE U XOY RAVNI:

$$x = r\cos\theta, \quad y = r\sin\theta$$

DAKLE, $x = \overbrace{\rho\cdot\sin\phi}^{r}\cdot\cos\theta$, $y = \underbrace{\rho\sin\phi}_{r}\sin\theta$,

ODNOSNO, ZA SFERNE KOORDINATE VAŽI SLEDEĆA VEZA:

$$\boxed{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\,\text{sin}\,\theta \\ z &= \rho\cos\text{FI} \end{aligned} \quad , \quad \begin{aligned} &\rho \geq 0 \\ &0 \leq \theta \leq 2\pi \\ &0 \leq \phi \leq \pi \end{aligned}}$$

ZA JAKOBIJAN VAŽI

$$J = \begin{vmatrix} x'_\rho & x'_\theta & x'_\phi \\ y'_\rho & y'_\theta & y'_\phi \\ z'_\rho & z'_\theta & z'_\phi \end{vmatrix} = \ldots = -\rho^2\sin\phi$$

$$\Downarrow$$

$$\boxed{|J| = \rho^2\sin\phi}$$

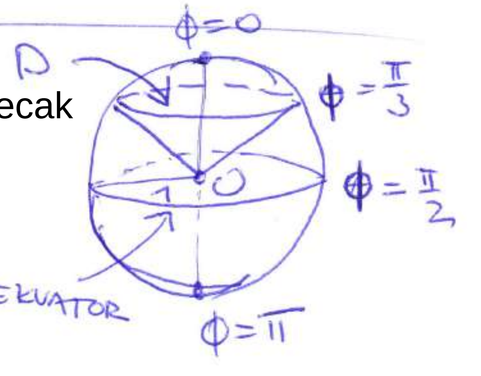
$$\| $$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right|$$

PRIMER

ANALOGIJA SA POLOŽAJEM TAČKE NA ZEMLJI.

loptin isecak




$\theta$ – GEOGRAFSKA DUŽINA (LONGITUDA)

$\phi$ – GEOG. ŠIRINA (LATITUDA)

(P9)

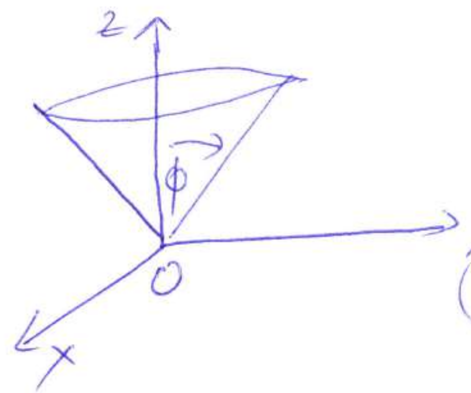
$$D^*: \begin{cases} 0 \le \rho \le 1 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{3} \end{cases}$$

- JEDNAČINA CENTRALNE SFERE POLUPREČNIKA a JE

$$\rho = a \quad \text{ILI} \quad D^* = \{(\rho, \theta, \phi) \mid \rho = a,\ \theta \in [0, 2\pi],\ \phi [0, \pi]\}$$

- $\phi = \frac{\pi}{2} \Rightarrow xOy$ RAVAN

- $\phi = \frac{\pi}{4}$



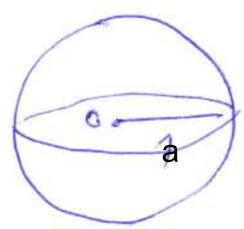

KONUS

PRIMER IZRAČUNATI ZAPREMINU LOPTE POLUPREČNIKA a.

Rš

$$ZAP = \iiint_D dx\,dy\,dz = \int_0^a d\rho \int_0^{2\pi} d\theta \int_0^{\pi} \rho^2 \sin\phi \, d\phi =$$

$$D^* = \{(\rho, \theta, \phi) \mid 0 \le \rho \le a,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$$




$$|J| = \rho^2 \sin\phi$$

$$= \int_0^a \rho^2 d\rho \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi = \int_0^a \rho^2 d\rho \int_0^{2\pi} d\theta \underbrace{(-\cos\phi)\Big|_0^{\pi}}_{-(-1-1)=2} = \int_0^a \rho^2 d\rho \; 2 \cdot \theta \Big|_0^{2\pi} =$$

$$= 2 \int_0^a \rho^2 d\rho \, (2\pi - 0) = 4\pi \frac{\rho^3}{3}\Big|_0^a = \frac{4a^3\pi}{3}$$

P10

$$\boxed{\text{ZAP} = \frac{4}{3} a^3 \pi}$$