(*) VEKTORSKA I SKALARNA POLJA. ELEMENTI TEORIJE POLJA

FUNKCIJU

$f : \mathbb{R}^n \to \mathbb{R}^m$, GDE $n, m \in \mathbb{N}$, MOŽEMO

KLASIFIKOVATI NA SLEDEĆI NAČIN:

1° AKO JE $m = 1 \to f$ JE SKALARNA FUNKCIJA

PRIMER: $f(x,y,z) = xyz - x^2$

2° AKO JE $m \geq 2 \to f$ JE VEKTORSKA FUNKCIJA

PRIMER: $f(x,y,z) = (x, -2xy, xz^2)$, OBIČNO PIŠEMO $\vec{f}(x,y,z) =$ —— " ——

3° AKO JE $n = 1 \to f$ FUNKCIJA JEDNE PROMENLJIVE

4° AKO JE $n \geq 1 \to f$ JE FUNK. VIŠE PROM.

- SVAKA VEKTORSKA FUNKCIJA VIŠE PROMENLJIVIH DEFINIŠE VEKTORSKO POLJE.

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j} = (P, Q),$$

GDE ~~SU~~ $P, Q : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ SKALARNE FUNKCIJE, JE VEKTORSKO POLJE. SLIČNO,

$$\vec{H}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

JE VEKTORSKO POLJE, $\vec{H} : D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$.

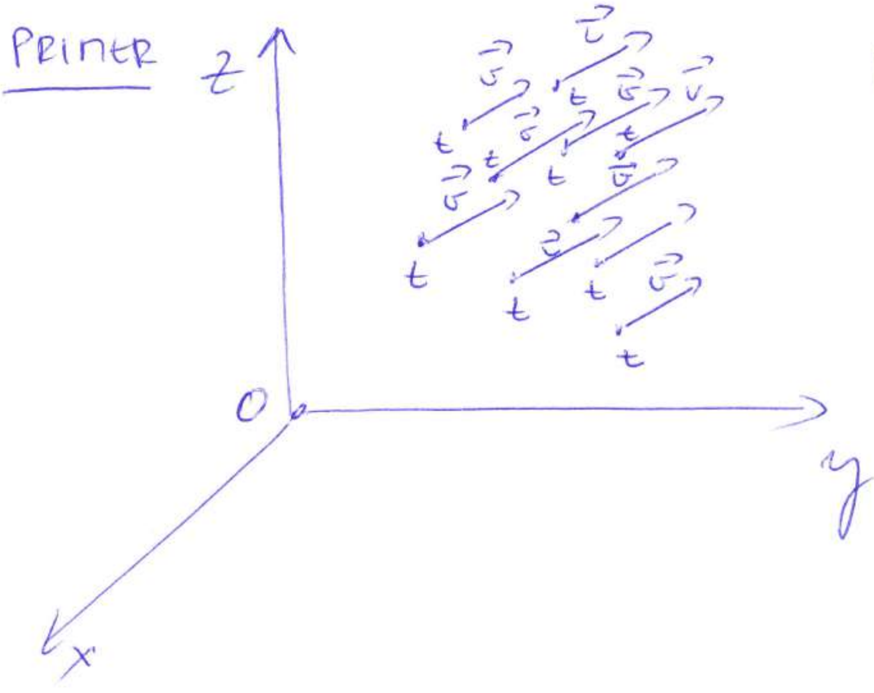
D - DOMEN POLJA

- SVAKA SKALARNA FUNKCIJA VIŠE PROMENLJIVIH JE SKALARNO POLJE.

~~f(x,y)~~ $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, ~~f~~ $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$

$f(x,y)$, $h(x,y,z)$

PRIMER



$\mathbb{R}^3$

(P2)

ČESTICE U PROSTORU

$t = t(x, y, z)$

t - TEMPERATURA ČESTICE, SKALARNO POLJE

~~$\vec{F}(x,y)$~~

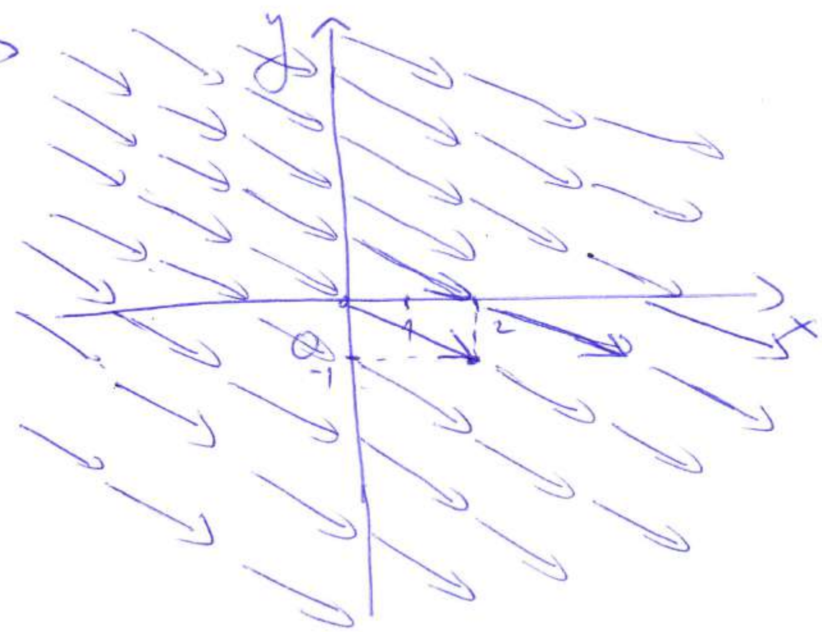
$\vec{v} = \vec{v}(x, y, z)$

$\vec{v}$ - BRZINA ~~ČESTICE~~ ČESTICE

$\vec{v}$ VEKTORSKO POLJE

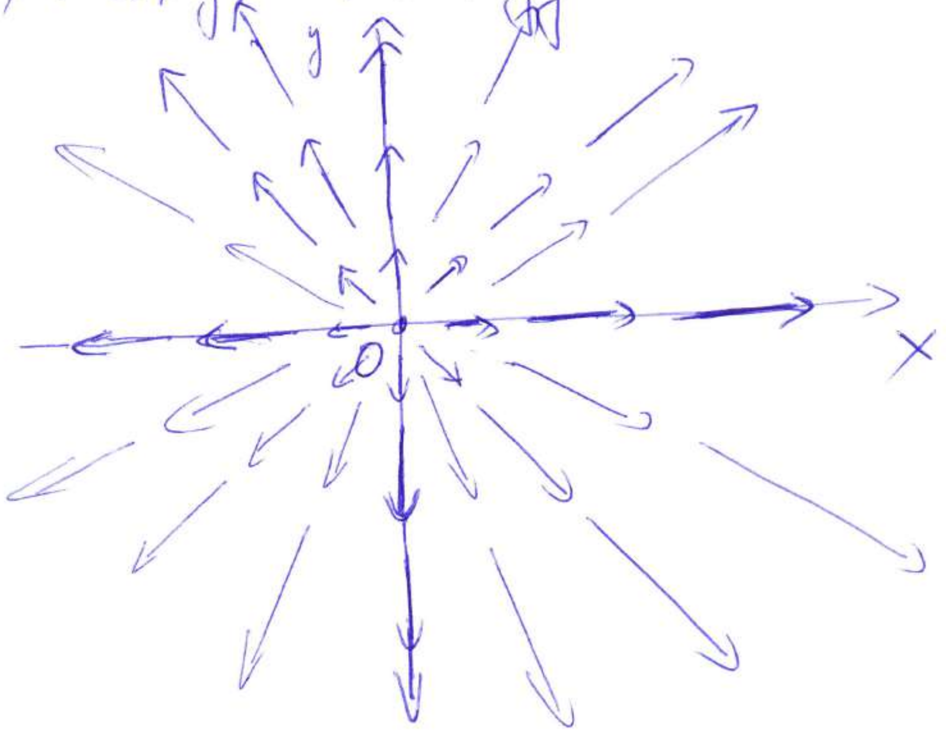
PRIMER PRIKAZATI GRAFIČKI SLEDEĆA VEKTORSKA POLJA

a) $\vec{F}(x, y) = 2\vec{i} - \vec{j}$



P3

b) $\vec{F}(x,y) = x\vec{i} + y\vec{j}$



GRADIJENT, DIVERGENCIJA, ROTOR

DEF. ZA DATO SKALARNO POLJE $u = f(x,y,z)$ ZA KOJE SU DEFINISANE VREDNOSTI $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$, VEKTOR

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

NAZIVA SE GRADIJENT FUNKCIJE f. OBELEŽAVA SE SA

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$

$$\operatorname{grad} f(M_0) = \left(\frac{\partial f}{\partial x}(M_0), \frac{\partial f}{\partial y}(M_0), \frac{\partial f}{\partial z}(M_0)\right)$$ JE

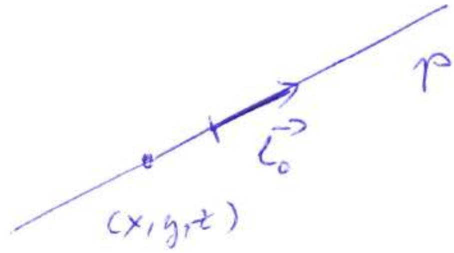
GRADIJENT U TAČKI $M_0 = (x_0, y_0, z_0)$. GRADIJENT JE LOKALNI OPERATOR, RAČUNA SE UVEK U ODNOSU NA NEKU TAČKU.

• ~~IZVOD U PRAVCU FUL IZVOD FUNKCI~~

DEF. IZVOD FUNKCIJE $u = f(x,y,z)$ U PRAVCU JEDINIČNOG VEKTORA $\vec{l_0} = (l_1, l_2, l_3)$ U TAČKI (x,y,z) SE DEFINIŠE KAO

$$\frac{\partial f}{\partial \vec{l_0}} \overset{\text{def.}}{=} \lim_{\Delta t \to 0} \frac{f((x,y,z) + \Delta t \cdot \vec{l_0})}{\Delta t} =$$

$$= \lim_{\Delta t \to 0} \frac{f(x + \Delta t l_1, y + \Delta t l_2, z + \Delta t l_3)}{\Delta t} .$$



IZ DEFINICIJE SLEDI DA JE

$$\frac{\partial f}{\partial \vec{l_0}} = \left. \frac{d f(x + l_1 \cdot t, y + l_2 \cdot t, z + l_3 \cdot t)}{dt} \right|_{t=0} =$$

$$= \frac{\partial f}{\partial x} \cdot (x + l_1 \cdot t)'_t + \frac{\partial f}{\partial y} (y + l_2 \cdot t)'_t + \frac{\partial f}{\partial z} (z + l_3 \cdot t)'_t =$$

$$= \frac{\partial f}{\partial x} \cdot l_1 + \frac{\partial f}{\partial y} l_2 + \frac{\partial f}{\partial z} l_3 = \operatorname{grad} f \underset{\substack{\uparrow \\ \text{SKALARNI} \\ \text{PROIZVOD}}}{\cdot} (l_1, l_2, l_3),$$

DAKLE,

$$\boxed{\frac{\partial f}{\partial \vec{l_0}}(M_0) = \operatorname{grad} f(M_0) \cdot \vec{l_0}, \text{ GDE JE } M_0 = (x_0, y_0, z_0), \; |\vec{l_0}| = 1 .}$$

- $$\frac{\partial f}{\partial \vec{l_0}} = \operatorname{grad} f \cdot \vec{l_0} = |\operatorname{grad} f| \cdot \overset{1}{\overset{\shortparallel}{|\vec{l_0}|}} \cdot \cos \measuredangle (\operatorname{grad} f, \vec{l_0})$$

$$= \operatorname{grad} f \cdot \cos \measuredangle (\operatorname{grad} f, \vec{l_0}), \text{ DAKLE}$$

$$\left| \frac{\partial f}{\partial \vec{l_0}} \right| = |\operatorname{grad} f| \cdot |\cos \measuredangle (\operatorname{grad} f, \vec{l_0})| \qquad -1 \le \cos \le 1$$

$$\operatorname{grad} f \parallel \vec{l_0} \quad \Rightarrow \quad |\cos\sphericalangle(\operatorname{grad} f, \vec{l_0})| = 1$$

IZVOD PO PRAVCU JE NAJVEĆI U PRAVCU GRADIJENTA FUNKCIJE. PRAVAC GRADIJENTA JE PRAVAC NAJBRŽE PROMENE (RASTA ILI OPADANJA) FUNKCIJE.

$$\sphericalangle(\operatorname{grad} f, \vec{l_0}) = 0^\circ \qquad\qquad \sphericalangle(\operatorname{grad} f, \vec{l_0}) = 180^\circ$$

$\vec{l_0}$ $\quad$ $\operatorname{grad} f$ $\qquad\qquad$ $\operatorname{grad} f$ $\quad$ $\vec{l_0}$ $\quad \left(\vec{l_0} = -\frac{\operatorname{grad} f}{|\operatorname{grad} f|}\right)$

$\cos = 1$ $\qquad\qquad$ $\cos = -1$

$$\frac{\partial f}{\partial \vec{l_0}} = |\operatorname{grad} f| \qquad\qquad \frac{\partial f}{\partial \vec{l_0}} = -|\operatorname{grad} f|$$

$$\vec{l_0} = \frac{\operatorname{grad} f}{|\operatorname{grad} f|}$$

$$\max_{\vec{l_0}} \left|\frac{\partial f}{\partial \vec{l_0}}\right| = |\operatorname{grad} f|$$

DEF <u>DIVERGENCIJA VEKTORSKOG POLJA</u>

$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ JE

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

PRESLIKAVA ~~V.P. U S.P.~~ VEKT. POLJE U SKALARNO POLJE.

DEF <u>ROTOR VEKT. POLJA</u> $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$

SE DEFINIŠE KAO

$$\operatorname{rot} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

PRESLIKAVA VEKT. P. U VEKT. POLJE.

• OPERATOR NABLA (HAMILTONOV OP.)

$$\nabla \overset{\text{df.}}{=} \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

VAŽI:

1) $\operatorname{grad} f = \nabla f$

2) $\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$ ↑ SKALARN PROIZVOD

3) $\operatorname{rot} \vec{F} = \nabla \times \vec{F}$ ↑ VEKT. PROIZVOD

PRIMER.

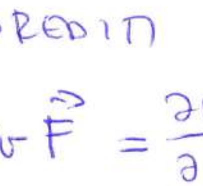
a) ODREDITI GRADIJENT POLJA $f(x,y,z) = xyz - 2y^2$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(yz, xz - 4y, xy\right)$$

b) ODREDITI $\operatorname{div}\vec{F}$ i $\operatorname{rot}\vec{F}$ ZA $\vec{F} = (x^2y, xyz, -x^2y^2)$

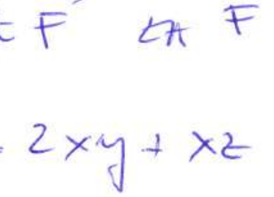
$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 2xy + xz$$

~~c)~~ $$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xyz & -x^2y^2 \end{vmatrix} = -(2x^2y + xy)\vec{i} + 2xy^2\vec{j} + (yz - x^2)\vec{k}.$$



$\operatorname{div}\vec{F}(M_0) < 0$

PONOR V.P.



$\operatorname{div}\vec{F}(M_0) > 0$

IZVOR V.P.

P7

- KLASIFIKACIJA VEKTORSKIH POLJA

1) V. P. $\vec{F}$ JE POTENCIJALNO (GRADIJENTNO ILI ~~KONZERVATIVNO~~ KONZERVATIVNO)

AKO POSTOJI SKALARNO POLJE f TAKVO DA JE

$$\vec{F} = \nabla f.$$

f JE POTENCIJAL POLJA $\vec{F}$.

2) V. P. $\vec{F}$ JE BEZVRTLOŽNO AKO JE $\operatorname{rot} \vec{F} = \vec{0}$.

3) V. P. $\vec{F}$ JE SOLENOIDNO (BEZIZVORNO) AKO JE

$$\operatorname{div} \vec{F} = 0.$$

(T) V. P. $\vec{F}$ JE POTENCIJALNO $\Longleftrightarrow$ $\vec{F}$ JE BEZVRTLOŽNO ($\operatorname{rot} \vec{F} = \vec{0}$).

PRIMER DOKAZATI DA JE V. P. $\vec{F} = (yz - 2y, xz - 2x, xy)$ GRADIJENTNO.

Reš

$$\operatorname{rot} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz-2y & xz-2x & xy \end{vmatrix} = \ldots = (0, 0, 0)$$

$f(x,y,z) = xyz - 2xy$ JE POTENCIJAL OD $\vec{F}$.

$$\nabla f = (yz - 2y, xz - 2x, xy)$$

(*) KRIVOLINIJSKI INTEGRAL VEKTORSKE FUNKCIJE (KRIV. INT. II VRSTE)

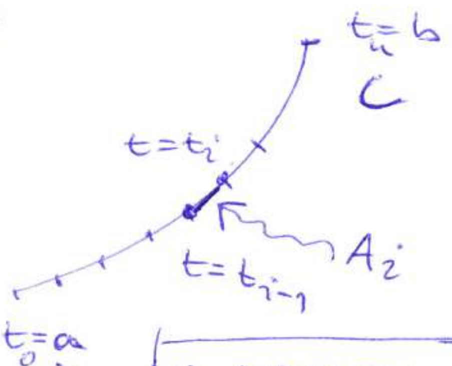
(PP)

POSMATRAJMO SILU KOJA JE U SVAKOJ TAČKI DEFINISANA VEKT. POLJEM $\vec{F} = (P(x,y), Q(x,y))$, ~~KOJA JE DEFINISAN~~ KOJI JE DEFINISAN NAD SKUPOM $G \subset \mathbb{R}^2$.

NEKA JE $C : \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, $t \in I = [a,b]$, DATA KRIVA I ~~DA~~ NEKA KRIVA C LEŽI U OBLASTI G.

SILA $\vec{F}$ VRŠI KRETANJE DUŽ KRIVE C I VRŠI ODREĐEN RAD.

PODELIMO KRIVU C NA SEGMENTE:



$t_0 = a < t_1 < t_2 < \dots < b = t_n$

$\Delta t_i = t_i - t_{i-1}$, $i = 1, 2, \dots, n$

$\Delta \vec{r}_i = \vec{r}(t_i) - \vec{r}(t_{i-1})$

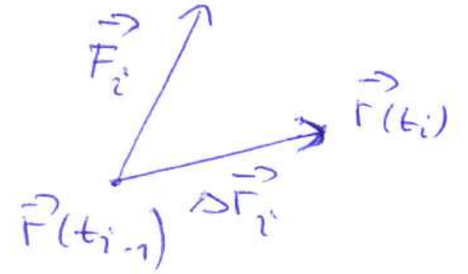
$\vec{F}_i = \left(P(x(t_i), y(t_i)), Q(x(t_i), y(t_i))\right)$

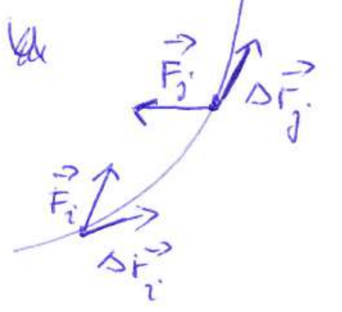
IZ FIZIKE:
$A = \vec{F} \cdot \vec{r}$
RAD = SILA · POMERANJE

A_i - RAD SILE NAD SEGMENTOM KRIVE ZA $t \in [t_{i-1}, t_i]$

$$A_i \approx F_i \cdot \Delta \vec{r}_i$$

SKALARNI PROIZVOD





UKUPAN RAD SILE $\vec{F}$ DUŽ KRIVE C:

$$A \approx \sum_{i=1}^{n} \vec{F}_i \cdot \Delta \vec{r}_i$$

DEF. KRIV. INT. VEKTORSKE FUNKCIJE $\vec{F}$

NAD KRIVOM C ~~SE DEFINIŠE KAO GR~~

JE GRANIČNA VREDNOST INTEGRALNE SUME

$$\int_C \vec{F}\cdot d\vec{r} \overset{df.}{=\!=} \lim_{\Delta t_i \to 0} \sum_{i=1}^{n} \vec{F}_i \cdot \Delta\vec{r}_i \,.$$

OSOBINE:

1) $\int_C (\alpha\vec{F} + \beta\vec{G})\,d\vec{r} = \alpha\int_C \vec{F}\,d\vec{r} + \beta\int_C \vec{G}\,d\vec{r}, \quad \alpha,\beta \in \mathbb{R}$

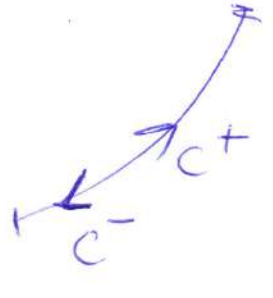
2) $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\,d\vec{r} + \int_{C_2} \vec{F}\,d\vec{r}, \quad C = C_1 \cup C_2,$

$C_1 \cap C_2 = \emptyset.$

3) SMER KRETANJA UTIČE NA VREDNOST KRIV. INT.

$$\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r}$$

JASNO: $\vec{F}\cdot(-d\vec{r}) = -(\vec{F}\cdot d\vec{r})$.



- PROMENA ORIJENTACIJE KRIVE

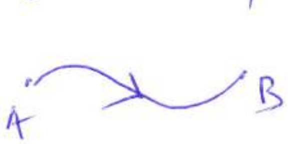
$x = x(t)$
$y = y(t) \quad t_0 \le t \le t_1$
$z = z(t)$

$\boxed{v = -t} \rightsquigarrow$

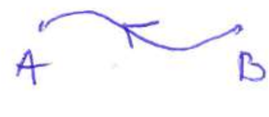
$-t_1 \le -t \le -t_0$

$-t_1 \le v \le -t_0$

$\vec{r}(v) = (x(-v), y(-v), z(-v))$



$\vec{r}(t) = (x(t), y(t), z(t))$



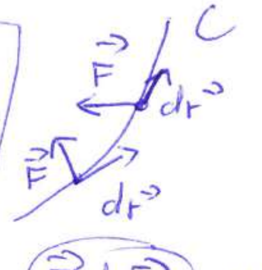
- IZRAČUNAVANJE KRIV. INT. VEKTORSKE FUNKCIJE (P10)

$$C:\ \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j},\quad t\in[a,b]$$

$$d\vec{r} = (dx, dy) = (x'(t)dt,\ y'(t)dt) = \vec{r}\,'(t)\,dt$$

ODNOSNO, ZAMENJUJUĆI U INTEGRAL, DOBIJAMO

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t))\cdot \vec{r}\,'(t)\,dt}$$



$\vec{F}\cdot d\vec{r} = A$

OPŠTI EL. INTEGRACIJE

VEZA IZMEĐU KRIV. INT. VEKTORSKE FUNKCIJE I KRIV. INT. SKALARNE FUNKCIJE:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\underbrace{\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}_{\vec{T}(t)}\cdot\underbrace{|\vec{r}\,'(t)|\,dt}_{ds}$$

$$= \int_C \vec{F}\cdot\vec{T}(t)\cdot ds$$

ds ← DIFERENCIJAL LUKA

$\vec{T}(t)$ ↑ JEDINIČNI TANGENTNI VEKTOR

- ~~ALT~~ DRUGA FORMULACIJA KRIV. INT. VEKT. FUNKCIJE:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (P, Q)\cdot(dx, dy) = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

ALTERNATIVNI NAZIV: <u>KRIV. INT. PO KOORDINATAMA</u>

- UOPŠTENJE NA SLUČAJ PROSTORNE KRIVE

$$C:\ \vec{r}(t) = (x(t), y(t), z(t)),\quad t\in[a,b]$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t), z(t))\cdot \vec{r}\,'(t)\,dt$$