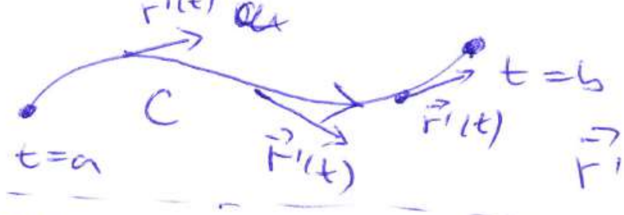
(*) IZRAČUNAVANJE KRIV. INT. VEKTORSKE FUNKCIJE (P1)

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t), z(t))\cdot \vec{r}\,'(t)\,dt,$$

GDE JE $C: \vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\cdot\vec{k}$, $t\in[a,b]$



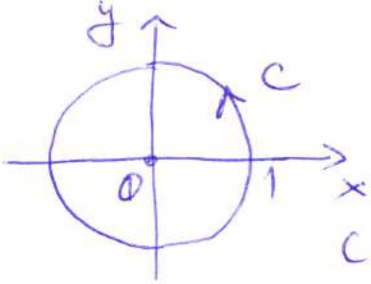
$\vec{r}\,'(t)$ JE TANGENTNI VEKTOR NA KRIVU C U TAČKI $\vec{r}(t)$.

• INTERPRETACIJA

$A = \int_C \vec{F}\cdot d\vec{r}$ - RAD SILE $\vec{F}$ DUŽ KRIVE C

<u>PRIMER</u> AKO JE C CENTRALNA JEDINIČNA KRUŽNICA, ORIJ. POZ., IZRAČUNATI RAD SILE $\vec{F} = x\,\vec{i} + y\,\vec{j}$ DUŽ TE KRIVE.

<u>REŠ.</u>



$x = \cos t$
$y = \sin t$ $\quad 0\le t\le 2\pi$

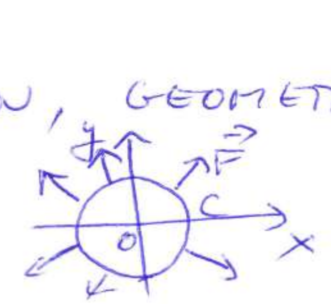
$C: \vec{r}(t) = (\cos t, \sin t)$, $\quad t\in[0,2\pi]$

$$A = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (\cos t, \sin t)\cdot \vec{r}\,'(t)\,dt =$$

$$= \int_0^{2\pi} (\cos t, \sin t)\cdot(-\sin t, \cos t)\,dt = \int_0^{2\pi}\left(\cancel{-\cos t\sin t} + \cancel{\sin t\cdot\cos t}\right)dt$$

$$= \int_0^{2\pi} 0\cdot dt = 0$$

II NAČIN, GEOMETRIJSKIM OPAŽANJEM, ~~IZRAČUNATO~~ MOŽEMO ZAKLJUČITI DA JE VEK. POLJE $\vec{F}$ ORTOGONALNO NA TANG. VEKTORE KRIVE, ODNOSNO $\vec{F}\cdot\vec{r}\,'(t)=0$



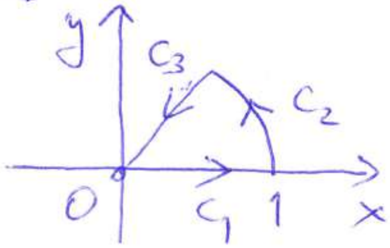
P2

$$\vec{F}(t) \perp \vec{r}\,'(t) \Rightarrow \vec{F}(t)\cdot \vec{r}\,'(t) = 0 \Rightarrow \int_0^{2\pi} \underbrace{\vec{F}\cdot \vec{r}\,'(t)}_{=0}\, dt = 0$$

<u>PRIMER</u> IZRAČUNATI $\int_C \vec{F}\cdot d\vec{r}$, AKO JE $\vec{F} = (y, x)$,

A C JE RUB ISEČKA CENTRALNE JEDINIČNE KRUŽNICE ZA $\Theta \in [0, \frac{\pi}{4}]$, POZITIVNO ORIJENTISANA.

<u>REŠ.</u>



$$C = C_1 \cup C_2 \cup C_3$$

$$C_1 : \vec{r}(t) = (t, 0) \Rightarrow d\vec{r} = \vec{r}\,'(t)\,dt$$
$$t \in [0,1] \qquad d\vec{r} = (1,0)\,dt$$

$$C_2 : \vec{r}(t) = (\cos t, \sin t)$$
$$t \in [0, \tfrac{\pi}{4}] \Rightarrow d\vec{r} = \vec{r}\,'(t)\,dt$$
$$d\vec{r} = (-\sin t, \cos t)\,dt$$

$$-C_3 : \vec{r}(t) = (t, t) \Rightarrow d\vec{r} = \vec{r}\,'(t)\,dt$$
$$t \in [0, \tfrac{\sqrt{2}}{2}] \qquad d\vec{r} = (1,1)\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^1 (y,x)\, \vec{r}\,'(t)\,dt = \int_0^1 (0,t)\,(1,0)\,dt = \int_0^1 (0+0)\,dt = 0$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_2} (y,x)\cdot d\vec{r} = \int_0^{\frac{\pi}{4}} (\sin t, \cos t)\cdot(-\sin t, \cos t)\,dt =$$

$$= \int_0^{\frac{\pi}{4}} (\cos^2 t - \sin^2 t)\,dt = \int_0^{\frac{\pi}{4}} \cos 2t\,dt = \frac{1}{2}\sin 2t \Big|_0^{\frac{\pi}{4}} = \frac{1}{2}$$

P3

$$\int_{C_3} \vec{F}\cdot d\vec{r} = -\int_{-C_3} \vec{F}\cdot d\vec{r} = -\int_0^{\frac{\sqrt{2}}{2}} (t,t)\cdot(1,1)\,dt =$$

$$= -\int_0^{\frac{\sqrt{2}}{2}} (t+t)\,dt = -2\int_0^{\frac{\sqrt{2}}{2}} t\,dt = -\cancel{2}\cdot\frac{t^2}{\cancel{2}}\Big|_0^{\frac{\sqrt{2}}{2}} = \frac{-2}{4} - 0 = -\frac{1}{2}$$

$$\int_C \vec{F}\cdot d\vec{r} = 0 + \frac{1}{2} - \frac{1}{2} = 0$$

(*) KRIVOLINIJSKI INTEGRAL GRADIJENTNOG (POTENCIJALNOG) POLJA

- $\vec{F}$ JE GRADIJENTNO, AKO POSTOJI SKALARNO POLJE f, TAKVO DA JE $\nabla f = \vec{F}$.

(T) (FUNDAMENTALNA TEOREMA ZA KRIV. INTEGRAL)
AKO JE $C: \vec{r} = \vec{r}(t)$, $t \in [a,b]$ GLATKA KRIVA SA POČETNOM TAČKOM P_1 I KRAJNJOM TAČKOM P_2, I $\vec{F} = \nabla f$ NEPREKIDNO GRADIJENTNO VEKTORSKO POLJE SKALARNE FUNKCIJE f, TADA JE

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(P_2) - f(P_1).$$

- DRUGAČIJI ZAPIS: $\int_C \nabla f \cdot d\vec{r} = \int_C \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\cdot(dx, dy, dz) =$

$$= \int_C \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \int_C df = f(P_2) - f(P_1)$$

DOKAZ TEOREME

P4

$$d\vec{r} = (dx, dy, dz) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \cdot dt,$$

ZNAMO DA JE $\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$,

SADA IMAMO

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = \int_a^b \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \cdot dt =$$

$$= \int_a^b \left(\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}\right) dt =$$

$$= \int_a^b \frac{df}{dt} \cdot dt = \int_a^b \frac{d}{dt}\left(f(\vec{r}(t))\right) dt = f(\vec{r}(t))\Big|_a^b =$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) = f(P_2) - f(P_1)$$

PRIMER IZRAČUNATI KRIV. INT. $\int_C \vec{F} \cdot d\vec{r}$ DUŽ PROIZVOLJNE KRIVE C KOJA SPAJA TAČKE $A(1,0)$ I $B(2,2)$, AKO JE
$\vec{F} = (2xy - 3, x^2)$.

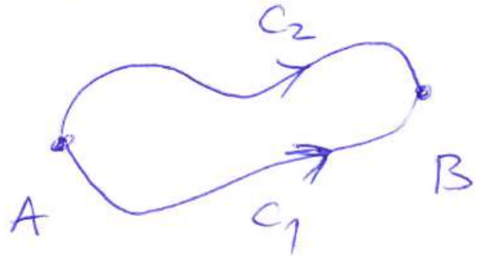
REŠ. PRIMETIMO DA JE ZA $f(x,y) = x^2 y - 3x$, VAŽI:
$\nabla f = (2xy - 3, x^2) = \vec{F}$, DAKLE $\vec{F}$ JE GRADIJENTNO V.P.
NA OSNOVU TEOREME

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \, d\vec{r} = f(B) - f(A) = 2 + 3 = 5$$ ***********

$f(2,2) = 2^2 \cdot 2 - 3 \cdot 2 = 2$
$f(1,0) = 0 - 3 = -3$

• OSOBINE KRIV. INT. GRADIJENTNOG POLJA: (P5)
$\vec{F} = \nabla f$

1) INT. NE ZAVISI OD IZBORA PUTANJE INTEGRACIJE

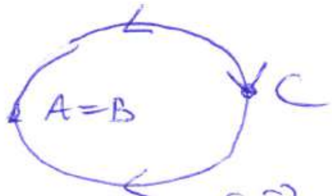


$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

2) ZA SVAKU ZATVORENU PUTANJU C VAŽI:

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

„CIRKULACIJA POLJA $\vec{F}$ DUŽ C JE NULA".



$\oint_C \vec{F} \cdot d\vec{r}$ ← CIRKULACIJA POLJA (=) RAD ~~ZA~~ DUŽ ZAT. PUTANJE

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \nabla f \, d\vec{r} = f(A) - f(A) = 0$$

• DEF. VEKTORSKO POLJE $\vec{F}$ JE KONZERVATIVNO, AKO ZA SVAKU ZATVORENU KRIVU C VAŽI:

$$\oint_C \vec{F} \cdot d\vec{r} = 0.$$

CIRKULACIJA U KONZERVATIVNOM POLJU JE UVEK NULA, TAKVA POLJA ~~SU~~: GRAVITACIONA I ELEKTRIČNA. MAGNETNO POLJE NIJE KONZERVATIVNO.

(*) NEZAVISNOST INTEGRALA OD PUTANJE INTEGRACIJE. (P6)
POTENCIJAL GRADIJENTNOG POLJA.

- SLEDEĆA TVRĐENJA SU EKVIVALENTNA

1) U. P. $\vec{F}$ JE GRADIJENTNO, TJ. POSTOJI SKALARNO POLJE f TAKVO DA JE $\vec{F} = \nabla f$

2) U. P. $\vec{F}$ JE KONZERVATIVNO, TJ. ZA SVAKU ZATVORENU KRIVU C VAŽI DA JE $\oint_C \vec{F} \cdot d\vec{r} = 0$.

3) $\int_C \vec{F} \cdot d\vec{r}$ NE ZAVISI OD IZBORA PUTANJE INTEGRACIJE.

4) V. P. $\vec{F}$ JE BEZVRTLOŽNO, TJ. $\text{rot}\,\vec{F} = \vec{0}$.

- KAKO PROVERITI DA LI JE $\vec{F}$ GRADIJENTNO POLJE?

1) $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$

$$\text{rot}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \vec{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \vec{k}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$\boxed{\text{rot}\,\vec{F} = (0,0,0)} \Rightarrow \vec{F}$ JE GRADIJENTNO U. P.

2) $\vec{F}(x,y) = (P(x,y), Q(x,y)) \rightsquigarrow \vec{F} = (P, Q, 0)$

$$\text{rot}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = (0,0,0)$$

$$\Rightarrow \boxed{\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}}$$

PRIMER PROVERITI DA LI JE V.P.

$\vec{F} = (yz^2 - 3y,\ xz^2 - 3x,\ 2xyz)$ GRADIJENTNO.

REŠ.

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2-3y & xz^2-3x & 2xyz \end{vmatrix} = \vec{i}\left(2xz - 2xz\right) - \vec{j}\left(2yz - 2zy\right) + \vec{k}\left(\cancel{x} z^2 - 3 - z^2 + 3\right) = (0,0,0)$$

DAKLE, $\operatorname{rot}\vec{F} = (0,0,0)$

$\Downarrow$

$\vec{F}$ JE GRADIJENTNO V.P.

- ODREĐIVANJE POTENCIJALNE FUNKCIJE

1) ANTI-DIFERENCIRANJEM

PRIMER ODREDITI POTENCIJALNU FUNKCIJU ZA V.P. $\vec{F} = (2xy - 3y,\ x^2 - 3x)$.

REŠ. $P = 2xy - 3y$, $Q = x^2 - 3x$.

$$\frac{\partial Q}{\partial x} = 2x - 3, \quad \frac{\partial P}{\partial y} = 2x - 3 \implies \vec{F} \text{ JE GRADIJENTNO!}$$

POSTOJI $f(x,y)$, TAKVO DA JE $\nabla f = \vec{F}$.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (P, Q) = (2xy - 3y,\ x^2 - 3x)$$

$$\frac{\partial f}{\partial x} = 2xy - 3y \to f = \int (2xy - 3y)\,dx = 2\frac{x^2}{2}y - 3xy + \varphi(y)$$

$$f = x^2 y - 3xy + \varphi(y)$$

$$\frac{\partial f}{\partial y} = \cancel{x}^2 - 3x$$

$$\Rightarrow \left(x^2 y - 3xy + \varphi(y)\right)'_y = x^2 - 3x$$

$$\cancel{x^2} - \cancel{3x} + \varphi'(y) = \cancel{x^2} - \cancel{3x}$$

$\varphi'(y) = 0 \implies \varphi(y) = C$, DAKLE

$$f(x,y) = x^2 y - 3xy + \varphi(y) = x^2 y - 3xy + C$$

$$\boxed{f(x,y) = x^2 y - 3xy} \quad (C=0)$$

2) IZRAČUNAVANJE INT. VEKT. POLJA KORISTEĆI POGODNU PUTANJU

$\vec{F} = \nabla f$, $\vec{F}(x,y,z) = (P, Q, R)$

$f(x,y,z) = ?$ ← POTENCIJAL OD $\vec{F}$

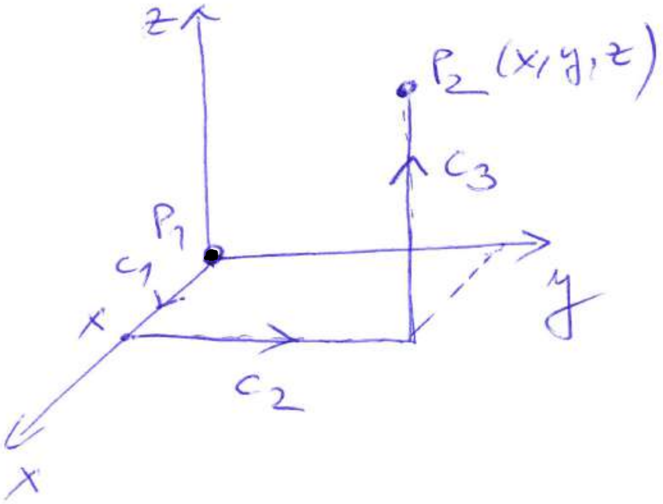
$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(P_2) - f(P_1)$$

FIKSIRAJMO TAČKU $P_1(x_0, y_0, z_0)$ $(= (0,0,0))$, A $P_2(x,y,z)$ NEKA BUDE PROMENLJIVA TAČKA:

$$\int_C \vec{F} \cdot d\vec{r} = f(x,y,z) - f(x_0, y_0, z_0)$$

$$f(x,y,z) = f(x_0, y_0, z_0) + \int_C \vec{F} \cdot d\vec{r}$$

C BIRAMO TAKO DA $C = C_1 \cup C_2 \cup C_3$, GDE SU C_1, C_2 I C_3 TRI PRAVOLINIJSKA SEGMENTA, TAKO DA SVAKI PARALELAN JEDNOJ OD KOORDINATNIH OSA:



$P_1 = (x_0, y_0, z_0) \; (= (0,0,0))$ (P9)

ZA P_1 SE NAJČEŠĆE UZIMA KOORDINATNI POČETAK.

TAD VAŽI:

$$C_1 : \vec{r}(t) = (t,0,0), \; t \in [0,x] \quad d\vec{r} = (1,0,0)dt$$

$$C_2 : \vec{r}(t) = (x,t,0), \; t \in [0,y] \quad d\vec{r} = (0,1,0)dt$$

$$C_3 : \vec{r}(t) = (x,y,t), \; t \in [0,z] \quad d\vec{r} = (0,0,1)dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (P,Q,R) \cdot d\vec{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} =$$

$$= \int_0^x P(t,0,0)\,dt + \int_0^y Q(x,t,0)\,dt + \int_0^z R(x,y,t)\,dt$$

ODNOSNO ZA POTENCIJAL POLJA $\vec{F}$ VAŽI

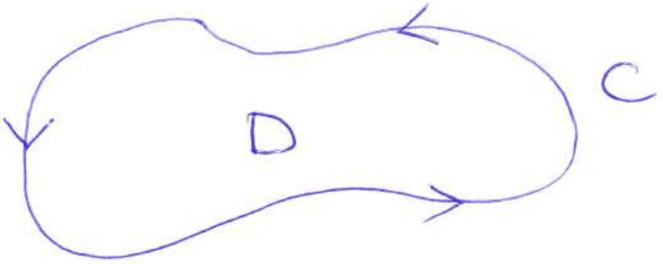
$$f(x,y,z) = \underbrace{f(x_0,y_0,z_0)}_{=0} + \int_0^x P(t,0,0)\,dt + \int_0^y Q(x,t,0)\,dt + \int_0^z R(x,y,t)\,dt$$

PRIMETIMO DA INT. $\int_C \vec{F} d\vec{r}$ MOŽEMO IZRAČUNATI PO BILO KOJOJ PUTANJI C KOJA SPAJA TAČKE P_1 I P_2.

⊛ GRINOVA TEOREMA

AKO $\vec{F}$ NIJE GRADIJENTNO U.P., ONDA NE MORA BITI $\oint_C \vec{F} \cdot d\vec{r} = 0$.

(T) (GRINOVA TEOREMA) NEKA JE C ZATVORENA, PROSTA, GLATKA, RAVNA, POZITIVNO ORIJENTISANA KRIVA I NEKA JE $D \subset \mathbb{R}^2$ UNUTRAŠNJOST KRIVE C.



NEKA JE V.P. $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ DEFINISANO I DIFERENCIJABILNO NA D. TADA VAŽI:

$$\boxed{\oint_C \vec{F} d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy}.$$

- GRINOVA FORMULA DAJE VEZU IZMEĐU KRIVOLINIJSKOG INT. U.P. I DVOSTRUKOG INT.

- $$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \Big(\underbrace{\tfrac{-y}{2}}_{P}, \underbrace{\tfrac{x}{2}}_{Q} \Big) \cdot (dx, dy) = \iint_D \left(\frac{1}{2} + \frac{1}{2} \right) dx\,dy = \iint_D dx\,dy$$

$\vec{F} = \left(-\frac{1}{2}y, \frac{1}{2}x\right)$

DAKLE, $\frac{1}{2}\oint_C (-y, x)\, d\vec{r} = \iint_D dx\,dy = \text{POV}(D)$

POV. OBLASTI MOŽEMO RAČUNATI KAO KRIV. INTEGRAL!