1.2 Trostruki integrali

1. **Izračunati $I = \iiint\limits_V (x^2y+z)dxdydz$ ako je**

$$V = \{(x,y,z) \in \mathbb{R}^3 : -2 \leq x \leq 2, -3 \leq y \leq 1, 1 \leq z \leq 5\}.$$

Rešenje:

$$I = \iiint\limits_V (x^2y+z)dxdydz = \int_{-2}^{2} \left(\int_{-3}^{1} \left(\int_{1}^{5} \left(x^2y+z\right) dz \right) dy \right) dx$$

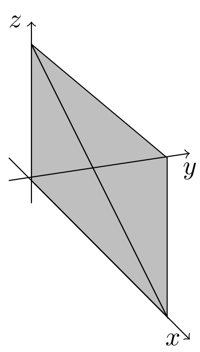
$$= \int_{-2}^{2} \left(\int_{-3}^{1} \left(x^2yz + \frac{z^2}{2} \right) \Bigg|_{z=1}^{z=5} dy \right) dx = \int_{-2}^{2} \left(\int_{-3}^{1} \left(4x^2y+12\right) dy \right) dx$$

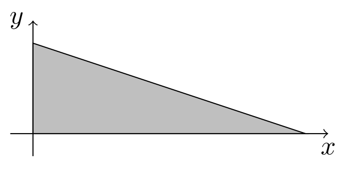
$$= \int_{-2}^{2} \left(4x^2\frac{y^2}{2} + 12y \right) \Bigg|_{y=-3}^{y=1} dx = \int_{-2}^{2} \left(-16x^2+48\right) dx$$

$$= \left(-16\frac{x^3}{3} + 48x \right) \Bigg|_{x=-2}^{x=2} = \frac{320}{3}.$$

2. **Izračunati zapreminu oblasti**

$$V = \{(x,y,z) \in \mathbb{R}^3 : 2x+3y+4z \leq 12, x \geq 0, y \geq 0, z \geq 0\}.$$





Rešenje:
Oblast V je ograničena sa četiri ravni u $\mathbb{R}^3$:

$$2x+3y+4z=12 \quad x=0 \quad y=0 \quad z=0.$$

Projekcija oblasti V na xy-ravan je trougao D u $\mathbb{R}^2$:

$$\begin{aligned} D &= \left\{(x,y)\in\mathbb{R}^2 : 2x+3y\le 12,\ x\ge 0,\ y\ge 0\right\} \\ &= \{(x,y)\in\mathbb{R}^2 : 0\le y\le \frac{1}{3}(12-2x),\ x\ge 0,\ 0\le x\le 6\} \end{aligned}$$

Tako se oblast V može zapisati na sledeći na sledeći način:

$$V=\{(x,y,z)\in\mathbb{R}^3 : 0\le z\le \frac{1}{4}(12-2x-3y),(x,y)\in D\}.$$

Zapremina oblasti V je

$$\begin{aligned} \Delta V &= \iiint\limits_V dxdydz = \iint\limits_D \left(\int\limits_0^{\frac{12-2x-3y}{4}} dz\right) dxdy = \iint\limits_D \frac{12-2x-3y}{4} dxdy \\ &= \int\limits_0^6 \left(\int\limits_0^{\frac{12-2x}{3}} \frac{12-2x-3y}{4} dy\right) dx = \int\limits_0^6 \left(3y-\frac{1}{2}xy-\frac{3}{4}\frac{y^2}{2}\right)\Bigg|_{y=0}^{y=\frac{12-2x}{3}} \\ &= -7\cdot\int\limits_0^6 \left(6-2x+\frac{x^2}{6}\right) dx = -7\cdot\left(6x-2\frac{x^2}{2}+\frac{x^3}{18}\right)\Bigg|_0^6 = ???. \end{aligned}$$

3. **Izračunati Jakobijan transformacije**

   (a) $\boldsymbol{x=\rho\cos\varphi,\ y=\rho\sin\varphi,\ z=z,}$
   $\boldsymbol{(\rho,\varphi,z)\in[0,\infty)\times[0,2\pi]\times\mathbb{R};}$

   (b) $\boldsymbol{x=\rho\cos\varphi\cos\theta,\ y=\rho\sin\varphi\cos\theta,\ z=\rho\sin\theta,}$
   $\boldsymbol{(\rho,\varphi,\theta)\in[0,\infty)\times[0,2\pi]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right].}$

   (c) $\boldsymbol{x=\rho\cos\varphi\sin\theta,\ y=\rho\sin\varphi\sin\theta,\ z=\rho\cos\theta,}$
   $\boldsymbol{(\rho,\varphi,\theta)\in[0,\infty)\times[0,2\pi]\times[0,\pi].}$

Rešenje:

(a) Jakobijan transformacije je:

$$J(\rho,\varphi,z)=\begin{vmatrix}\cos\varphi & -\rho\sin\varphi & 0\\ \sin\varphi & \rho\cos\varphi & 0\\ 0 & 0 & 1\end{vmatrix}=\rho\cos^2\varphi+\rho\sin^2\varphi=\rho.$$

(b) Jakobijan transformacije je:

$$\begin{aligned}
J(\rho,\varphi,\theta) &= \frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} \\
&= \begin{vmatrix} \cos\varphi\cos\theta & -\rho\sin\varphi\cos\theta & -\rho\cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\theta & 0 & \rho\cos\theta \end{vmatrix} \\
&= \sin\theta\left(\rho^2\sin^2\varphi\sin\theta\cos\theta + \rho^2\cos^2\varphi\sin\theta\cos\theta\right) \\
&- \rho\cos\theta\left(\rho\cos^2\varphi\cos^2\theta + \rho\sin^2\varphi\cos^2\theta\right) \\
&= \rho^2\sin^2\theta\cos\theta + \rho^2\cos^3\theta = \rho^2\cos\theta.
\end{aligned}$$

(c) Jakobijan transformacije je:

$$\begin{aligned}
J(\rho,\varphi,\theta) &= \frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} \\
&= \begin{vmatrix} \cos\varphi\sin\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\theta & 0 & -\rho\sin\theta \end{vmatrix} \\
&= \cos\theta\left(\rho^2\sin^2\varphi\sin\theta\cos\theta - \rho^2\cos^2\varphi\sin\theta\cos\theta\right) \\
&- \rho\sin\theta\left(\rho\cos^2\varphi\sin^2\theta + \rho\sin^2\varphi\sin^2\theta\right) \\
&= \rho^2\cos^2\theta\sin\theta - \rho^2\sin^3\theta = -\rho^2\sin\theta.
\end{aligned}$$

4. **Izračunati $\boldsymbol{I = \iiint\limits_V (x^2+y^2)dxdydz}$ ako je oblast integracije**

$$\boldsymbol{V = \{(x,y,z)\in\mathbb{R}^3 : 2 \le z \le 4 - x^2 - y^2\}.}$$

Rešenje: Oblast V je ograničena paraboloidom $z = 4 - x^2 + y^2$ sa temenom u $(0,0,4)$ i sa ravni $z = 2$.

Presek datog paraboloida i ravni je centralna kružnica poluprečnika $\sqrt{2}$ u ravni $z = 2$:

$$z = 4 - x^2 + y^2 \wedge z = 2 \Leftrightarrow 2 = 4 - x^2 + y^2 \wedge z = 2 \Leftrightarrow x^2 + y^2 = 2 \wedge z = 2.$$

Odatle je projekcija D oblasti V na xy-ravan krug poluprečnika $\sqrt{2}$:

$$D = \{(x,y)\in\mathbb{R}^2 : x^2 + y^2 \le 2\}.$$

Tako je traženi integral nad oblasti V :

$$\begin{aligned}
I &= \iiint\limits_V (x^2+y^2)dxdydz = \iint\limits_D \left(\int\limits_2^{4-x^2-y^2} (x^2+y^2)dz\right) dxdy \\
&= \iint\limits_D (2 - x^2 - y^2)(x^2+y^2)dxdy.
\end{aligned}$$

Uvodimo smenu polarnim koordinatama

$$x = \rho\cos\varphi,\ y = \rho\sin\varphi,\ \rho \in [0, \sqrt{2}],\ \varphi \in [0, 2\pi]$$

za koju važi da je $x^2 + y^2 = \rho^2$ i $J = \rho$. Sada gornji integral postaje:

$$I = \int_0^{2\pi}\left(\int_0^{\sqrt{2}}(2-\rho^2)\rho^3 d\rho\right)d\varphi = \int_0^{2\pi}d\varphi\int_0^{\sqrt{2}}\left(2\rho^3-\rho^5\right)d\rho$$

$$= \varphi\Big|_{\varphi=0}^{\varphi=2\pi}\left(2\frac{\rho^4}{4}-\frac{\rho^5}{5}\right)\Bigg|_{\rho=0}^{\rho=\sqrt{2}} = 2\pi\left(2-\frac{8}{6}\right) = \frac{4\pi}{3}.$$

5. **Izračunati zapreminu oblasti:**

(a) $\boldsymbol{V = \{(x,y,z)\in\mathbb{R}^3 : 1 \le x^2+y^2+z^2 \le 9\}}$**,**

(b) $\boldsymbol{V = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2 \le z^2, x^2+y^2+z^2 \le 9, z \le 0\}}$**.**

Rešenje:

(a) Zadata oblast ograničena je sa dve sfere. Uvodimo sferne koordinate:

$$\begin{aligned} x &= \rho\cos\varphi\cos\theta \\ y &= \rho\sin\varphi\cos\theta \\ z &= \rho\sin\theta \end{aligned}\qquad (\rho,\varphi,\theta)\in[1,3]\times[0,2\pi]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right].$$

Jakobijan transformacije je $|J| = \rho^2\cos\theta$. Onda je zapremina:

$$\Delta V = \iiint_V dxdydz = \int_0^{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_1^3\rho^2\cos\theta d\rho\right)d\theta\right)d\varphi$$

$$= \int_0^{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\rho^3}{3}\Bigg|_{\rho=1}^{\rho=3}\cos\theta d\theta\right)d\varphi$$

$$= \frac{26}{3}\int_0^{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta d\theta\right)d\varphi = \frac{26}{3}\int_0^{2\pi}\sin\theta\Bigg|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}d\varphi$$

$$= \frac{52}{3}\int_0^{2\pi}d\varphi = \frac{52}{3}\,\varphi\Big|_0^{2\pi} = \frac{104\pi}{3}.$$

(b) Oblast V predstavlja isečak lopte. Slično kao u prethodnom zadatku uvodimo sferne koordinate:

$$\begin{aligned} x &= \rho\cos\varphi\cos\theta \\ y &= \rho\sin\varphi\cos\theta \\ z &= \rho\sin\theta \end{aligned}\qquad (\rho,\varphi,\theta)\in[0,3]\times[0,2\pi]\times\left[-\frac{\pi}{2},-\frac{\pi}{4}\right].$$

Jakobijan transformacije je $|J| = \rho^2 \cos\theta$. Onda je zapremina:

$$\Delta V = \iiint\limits_V dxdydz = \int\limits_0^{2\pi} \left(\int\limits_{-\frac{\pi}{4}}^{-\frac{\pi}{2}} \left(\int\limits_0^3 \rho^2 \cos\theta d\rho \right) d\theta \right) d\varphi$$

$$= \int\limits_0^{2\pi} \left(\int\limits_{-\frac{\pi}{4}}^{-\frac{\pi}{2}} \frac{\rho^3}{3}\Big|_{\rho=0}^{\rho=3} \cos\theta d\theta \right) d\varphi = \int\limits_0^{2\pi} \left(\int\limits_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} 9\cos\theta d\theta \right) d\varphi$$

$$= \int\limits_0^{2\pi} \left(9\sin\theta\Big|_{\theta=-\frac{\pi}{2}}^{\theta=-\frac{\pi}{4}} \right) d\varphi = 9\left(2-\sqrt{2}\right)\pi.$$

6. **Odrediti težište homogenog tela:**

   (a) $V = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 2, 0 \le y \le 3, 0 \le z \le 4\}$,

   (b) $V = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 9, z \ge 0\}$.

Rešenje: Za homogeno telo je gustina konstantna u svakoj tački. Pretpostavićemo da je $\mu(x,y,z) = \mu$, $(x,y,z) \in V$.

(a) Masa je:

$$m = \iiint\limits_V \mu dxdydz = \mu \iiint\limits_V dxdydz = \mu \int\limits_0^2 \left(\int\limits_0^3 \left(\int\limits_0^4 dz \right) dy \right) dx$$

$$= \mu \int\limits_0^2 dx \cdot \int\limits_0^3 dy \cdot \int\limits_0^4 dz = \mu \cdot 2 \cdot 3 \cdot 4 = 24\mu.$$

Težište ploče $T(x_T, y_T, z_T)$ dobijamo određujući pojedinačno svaku koordinatu:

$$x_t = \frac{1}{m} \iiint\limits_V x\mu dxdydz = \frac{1}{24} \int\limits_0^4 \left(\int\limits_0^3 \left(\int\limits_0^2 xdx \right) dy \right) dz$$

$$= \frac{1}{24} \int\limits_0^4 \left(\int\limits_0^3 \frac{x^2}{2}\Big|_{x=0}^{x=2} dy \right) dz = \frac{1}{12} \int\limits_0^4 \left(\int\limits_0^3 dy \right) dz$$

$$= \frac{1}{12} \int\limits_0^4 dz \int\limits_0^3 dy = \frac{1}{12} \cdot 4 \cdot 3 = 1$$

$$y_t = \frac{1}{m}\iiint\limits_V y dxdydz = \frac{1}{24}\int\limits_0^2 \left(\int\limits_0^4 \left(\int\limits_0^3 y dy\right) dz\right) dx$$
$$= \frac{1}{24}\int\limits_0^2 \left(\int\limits_0^4 \frac{y^2}{2}\Bigg|_{y=0}^{y=3} dz\right) dx = \frac{9}{48}\int\limits_0^2 \left(\int\limits_0^4 dz\right) dx$$
$$= \frac{9}{48}\int\limits_0^2 dx \int\limits_0^4 dz = \frac{9}{48}\cdot 2\cdot 4 = \frac{3}{2},$$

$$z_t = \frac{1}{m}\iiint\limits_V z\mu dxdydz = \frac{1}{24}\int\limits_0^2 \left(\int\limits_0^3 \left(\int\limits_0^4 z dz\right) dy\right) dx$$
$$= \frac{1}{24}\int\limits_0^2 \left(\int\limits_0^3 \frac{z^2}{2}\Bigg|_{z=0}^{z=4} dy\right) dx$$
$$= \frac{1}{3}\int\limits_0^2 \left(\int\limits_0^3 dy\right) dx = \frac{1}{3}\int\limits_0^2 dx \int\limits_0^3 dy = \frac{1}{3\,\mu}\cdot 2\cdot 3 = 2\,\mu.$$

Dakle, masa tela je $m = 24\mu$, a težište $T\left(1, \frac{3}{2}, 2\right)$.

(b) Uvodimo sferne koordinate

$$\begin{array}{l} x = \rho\cos\varphi\cos\theta \\ y = \rho\sin\varphi\cos\theta \\ z = \rho\sin\theta \end{array} \qquad (\rho,\varphi,\theta) \in [0,3]\times[0,2\pi]\times\left[0,\frac{\pi}{2}\right].$$

Jakobijan smene je $|J| = \rho^2\cos\theta$.

$$m = \mu\int\limits_0^{2\pi}\left(\int\limits_0^3\left(\int\limits_0^{\frac{\pi}{2}} \rho^2\cos\theta d\theta\right) d\rho\right) d\varphi$$
$$= \mu\int\limits_0^{2\pi}\left(\int\limits_0^3 \rho^2\sin\theta\Bigg|_{\theta=0}^{\theta=\frac{\pi}{2}} d\rho\right) d\varphi$$
$$= \mu\int\limits_0^{2\pi}\left(\int\limits_0^3 \rho^2 d\rho\right) d\varphi = \mu\int\limits_0^{2\pi} d\varphi\int\limits_0^3 \rho^2 d\rho = 2\pi\mu\cdot\frac{\rho^3}{3}\Bigg|_{\rho=0}^{\rho=3} = 18\pi\mu.$$

$$x_t = \frac{1}{18\pi}\int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}}\left(\int_0^3 \rho\cos\theta\cos\varphi\cdot\rho^2\cos\theta d\rho\right)d\theta\right)d\varphi$$
$$= \frac{1}{18\pi}\int_0^3 \rho^3 d\rho\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta\int_0^{2\pi}\cos\varphi d\varphi$$
$$= \frac{1}{18\pi}\cdot\frac{\rho^4}{4}\Big|_{\rho=0}^{\rho=3}\int_0^{\frac{\pi}{2}}\frac{1+\cos(2\theta)}{2}d\theta\cdot\int_0^{2\pi}\cos\varphi d\varphi$$
$$= \frac{1}{18\pi}\cdot\frac{81}{4}\left(\frac{1}{2}\int_0^{\frac{\pi}{2}}d\theta+\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos(2\theta)d\theta\right)\sin\varphi|_{\varphi=0}^{\varphi=2\pi}=0.$$

$$y_t = \frac{1}{18\pi}\int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}}\left(\int_0^3 \rho\cos\theta\sin\varphi\cdot\rho^2\cos\theta d\rho\right)d\theta\right)d\varphi$$
$$= \frac{1}{18\pi}\int_0^3 \rho^3 d\rho\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta\int_0^{2\pi}\sin\varphi d\varphi$$
$$= \frac{1}{18\pi}\frac{\rho^4}{4}\Big|_{\rho=0}^{\rho=3}\int_0^{\frac{\pi}{2}}\frac{1+\cos(2\theta)}{2}d\theta\cdot\int_0^{2\pi}\sin\varphi d\varphi$$
$$= \frac{1}{18\pi}\cdot\frac{81}{4}\left(\frac{1}{2}\int_0^{\frac{\pi}{2}}d\theta+\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos(2\theta)d\theta\right)(-\cos\varphi)|_{\varphi=0}^{\varphi=2\pi}=0.$$

$$z_t = \frac{1}{18\pi}\int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}}\left(\int_0^3 \rho\sin\theta\cdot\rho^2\cos\theta d\rho\right)d\theta\right)d\varphi$$
$$= \frac{1}{18\pi}\int_0^3 \rho^3 d\rho\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta d\theta$$
$$= \frac{1}{18\pi}\cdot\frac{\rho^4}{4}\Big|_0^3\cdot\varphi|_{\varphi=0}^{\varphi=2\pi}\cdot\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta d\theta=\frac{1}{18\pi}\cdot\frac{81}{4}\cdot 2\pi\cdot\int_0^1 tdt$$
$$= \frac{9}{4}\cdot\frac{t^2}{2}\Big|_{t=0}^{t=1}=\frac{9}{8}.$$

Dakle, masa polulopte sa centrom u $(0,0,0)$ poluprečnika 3 i konstantne gustine μ je $m = 18\pi\mu$, dok težište ima koordinate $T\left(0,0,\frac{9}{8}\right)$.

7. **Izračunati $I = \iiint\limits_V (x^2 + y^2)dxdydz$, gde je**

$$V = \{(x,y,z) \in \mathbb{R}^3 :\ x^2 + y^2 \geq 2,\ x^2 + y^2 \leq 7 - z,\ z \geq 3\}.$$

Rešenje: Presek kružnog cilindra datog jednačinom $x^2 + y^2 = 2$ i rotacionog paraboloida datog sa $z = 7 - x^2 - y^2$ je kružnica

$$\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 2,\ z = 5\}.$$

Presek ravni date jednačinom $z = 3$ i paraboloida datog jednačinom $z = 7 - x^2 - y^2$ je kružnica

$$\{(x,y,z) \in \mathbb{R}^3 :\ x^2 + y^2 = 4,\ z = 3\}.$$

Projekcija oblasti V na xy-ravan je kružni prsten

$$D = \{(x,y) \in \mathbb{R}^2 :\ 2 \leq x^2 + y^2 \leq 4\}.$$

Uvođenjem polarnih koordinata dobijamo:

$$I = \iiint\limits_V (x^2 + y^2)dxdydz = \iint\limits_D \left(\int\limits_3^{7-x^2-y^2} (x^2 + y^2)dz \right) dxdy$$

$$= \iint\limits_{Proj_{xy}} (x^2 + y^2)(7 - x^2 - y^2 - 3)dxdy = \int\limits_0^{2\pi} d\varphi \int\limits_{\sqrt{2}}^{2} \rho^3(4 - \rho^2)d\rho$$

$$= 2\pi \left(\rho^4 - \frac{\rho^6}{6}\right)\Bigg|_{\rho=\sqrt{2}}^{\rho=2} = \frac{16\pi}{3}.$$