FUNKCIJE VIŠE PROMENLJIVIH

UVODNI POJMOVI

$z = f(x,y)$ FUNKCIJA OD DVE NEZAVISNE PROMENLJIVE x i y

$u = f(x,y,z)$ FUNK. OD TRI NEZAVISNE PROM. x, y, z.

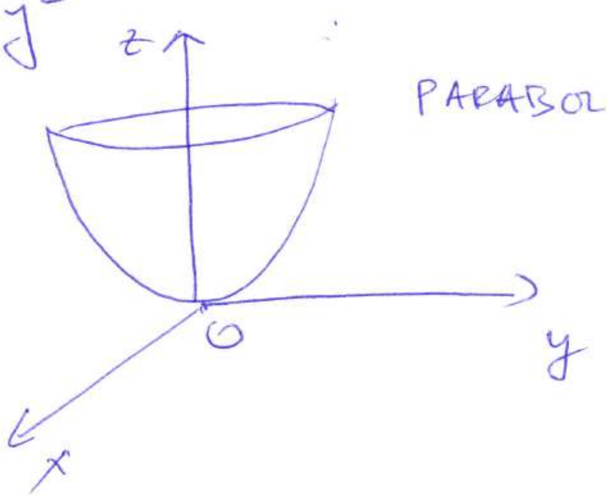
$(x,y) \in D \subseteq \mathbb{R}^2$ ILI $(x,y,z) \in D \subseteq \mathbb{R}^3$

D JE DOMEN FUNKCIJE

$f : D \to \mathbb{R}$, TJ. f JE SKALARNA FUNKCIJA

PRIMER PREDSTAVITI GRAFIČKI

a) $z = x^2 + y^2$

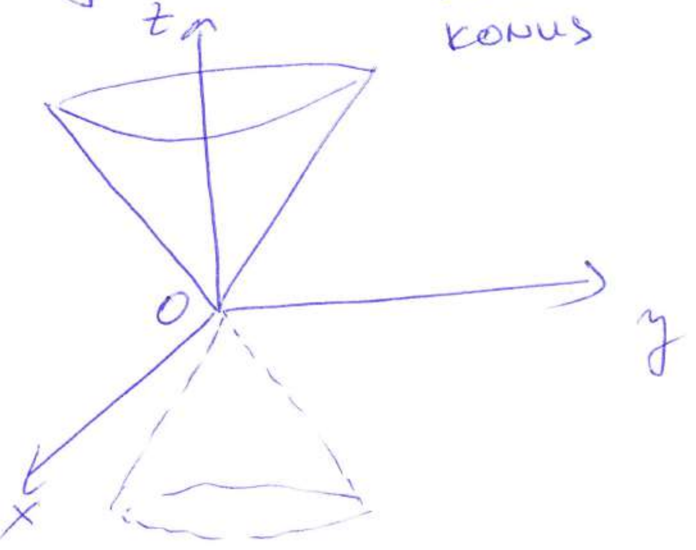


PARABOLOID

$y = 0 \Rightarrow z = x^2$

$x = 0 \Rightarrow z = y^2$

b) $z^2 = x^2 + y^2$



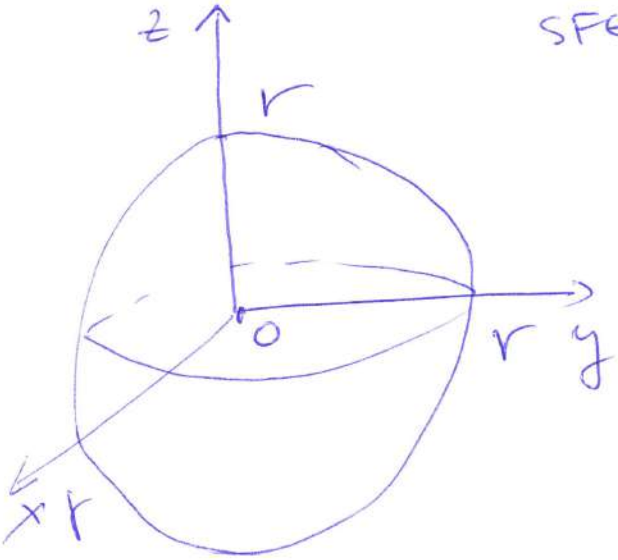
KRUŽNI KONUS

$x = 0 \Rightarrow z = |y|$

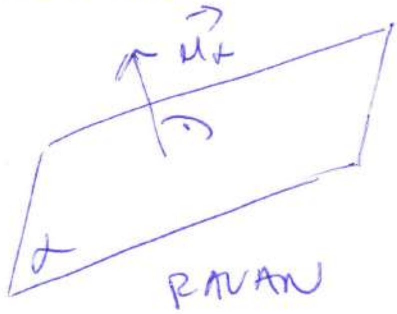
$y = 0 \Rightarrow z = |x|$

c) $z^2+x^2+y^2=r^2$

SFERA



d) $A\cdot x+B\cdot y+Cz+D=0$



$\vec{n}_\alpha=(A,B,C)$

RAVAN

(*) PARCIJALNI IZVODI

$z=f(x,y)\ ,\ D\subseteq\mathbb{R}^2$

DEF. PARCIJALNI IZVOD PO x-u FUNKCIJE $z=f(x,y)$ ~~u~~ TAČKI (x,y) SE DEFINIŠE NA SLEDEĆI NAČIN

$$\frac{\partial z}{\partial x}\left(\text{ili }\frac{\partial f}{\partial x}\right)\overset{\text{def.}}{=}\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x},$$

SLIČNO

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial y}\overset{\text{def.}}{=}\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}.$$

- $\Delta_x z=f(x+\Delta x,y)-f(x,y)$ ← PRIRAŠTAJ PO x-u
- $\Delta_y z=f(x,y+\Delta y)-f(x,y)$ ← PRIRAŠTAJ PO y-u
- $\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)$ ← TOTALNI PRIRAŠTAJ FUNKCIJE z

PRIMER. IZRAČUNATI

a) $\frac{\partial}{\partial x}\left(x^2 + xy - \sin y^2\right) = 2x + y$

b) $\frac{\partial}{\partial y}\left(x^2 + xy - \sin y^2\right) = x - \cos y^2 \cdot (2y)$

c) $\frac{\partial}{\partial x}\left(\ln(x^2 - xy) + 3xy\right) = \frac{1}{x^2 - xy}(2x - y) + 3y$

d) $\frac{\partial}{\partial y}\left(xy^3 - \frac{1}{2}e^{2y}\right) = 3xy^2 - \frac{1}{\cancel{2}}e^{2y} \cdot \cancel{2} =$

$= 3xy^2 - e^{2y}$

- AKO U OKOLINI TAČKE (x,y) FUNKCIJA $z = f(x,y)$ IMA NEPREKIDNE PARCIJALNE IZVODE $\frac{\partial f}{\partial x}$ I $\frac{\partial f}{\partial y}$, ONDA JE $z = f(x,y)$ DIFERENCIJABILNA U (x,y), A IZRAZ

$$\boxed{dz = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}$$

NAZIVAMO TOTALNI DIFERENCIJAL PRVOG REDA.

TADA VAŽI:

$$\Delta z \approx dz, \quad \text{GDE JE}$$

$\Delta z = f(x + \Delta x, y + \Delta y) - f(x,y)$ ~~PRIR~~ TOTALNI PRIRAŠTAJ FUNKCIJE f U TAČKI (x,y).

$$\Delta z = dz$$
$$\Delta x \to 0, \Delta y \to 0$$

- IZVOD SLOŽENE FUNKCIJE

a) $z = F(u, v)$, GDE SU $u = u(x,y)$ I $v = v(x,y)$

TADA VAŽI

$$\frac{\partial z}{\partial x} = \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x} \quad I$$

$$\frac{\partial z}{\partial y} = \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

b) NEKA JE $z = F(u, v)$, GDE SU $u = u(t)$ I $v = v(t)$.

TADA JE

$$z = F(u(t), v(t)) = F(t)$$

$$\frac{\partial z}{\partial t} = \frac{dz}{dt} = \frac{\partial F}{\partial u}\cdot\frac{du}{dt} + \frac{\partial F}{\partial v}\cdot\frac{dv}{dt}.$$

PRIMER. AKO JE $z = \ln(u^2+v)$, $u = e^{x+y^2}$, $v = x^2 - y$

ODREDITI

$$\frac{\partial z}{\partial x} = \frac{2u}{u^2+v}\cdot\frac{\partial u}{\partial x} + \frac{1}{u^2+v}\cdot\frac{\partial v}{\partial x} =$$

$$= \frac{2u}{u^2+v}\cdot e^{x+y^2} + \frac{2x}{u^2+v} = \frac{2e^{2x+2y^2}}{e^{2x+2y^2}+x^2-y} + \frac{2x}{e^{2x+2y^2}+x^2-y}$$

PRIMER AKO JE $z = \sin(u+2v)$, $u = t^2$, $v = 3t$, ODREDITI $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \cos(u+2v)\cdot 1\cdot\frac{du}{dt} + \cos(u+2v)\cdot 2\cdot\frac{dv}{dt} =$$

$$= \cos(u+2v)\cdot 2t + \cos(u+2v)\cdot 2\cdot 3 = \cos(u+2v)\cdot(6+2t)$$

- PARCIJALNI IZVODI VIŠEG REDA

NEKA JE $z = f(x,y)$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right), \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right),$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right), \quad \frac{\partial^2 z}{\partial^2 y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

(T) AKO SU FUNKCIJA $z = f(x,y)$ I NJENI PARCIJALNI IZVODI $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$ I $\frac{\partial^2 f}{\partial y \partial x}$ NEPREKIDNI U NEKOJ TAČKI $M(x,y)$, ONDA U TOJ TAČKI VAŽI

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

PRIMER AKO JE $f = x^2 y - 3xy$,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^2 - 3x\right) = 2x - 3$$

$$\|$$

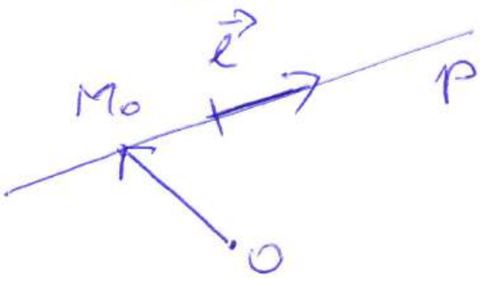
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy - 3y\right) = 2x - 3$$

(*) IZVOD PO PRAVCU

NEKA JE $\vec{e} = (a,b)$ JEDINIČNI VEKTOR, TJ. $|\vec{e}| = 1$

$|\vec{e}| = \sqrt{a^2+b^2}$

$z = f(x,y)$, $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$, $M_0(x_0,y_0) \in D$.



$\overrightarrow{OM_0} + t \cdot \vec{e} \in$ PRAVA p

$t \in \mathbb{R}$

$$\overrightarrow{OM} + t \cdot \vec{\ell} = (x_0, y_0) + t(a, b) = (x_0 + ta, y_0 + tb)$$

PARAMETARSKA JEDNAČINA PRAVE p

POSMATRAJMO FUNKCIJU OD JEDNE PROMENLJIVE, ODREĐENU ~~SA~~

$$z(t) = f(\underbrace{x_0 + ta}_{x(t)}, \underbrace{y_0 + tb}_{y(t)}).$$ SADA MOŽEMO DA

ODREDIMO IZVOD PO t-u:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \frac{\partial f}{\partial x} \cdot a + \frac{\partial f}{\partial y} \cdot b,$$

$\cancel{\frac{dz}{dt} = \frac{\partial f}{\partial x}(x(t)) \cdot a + \frac{\partial f}{\partial y}(y(t))}$ ODNOSNO

$$\frac{dz}{dt} = \left(\frac{\partial f}{\partial x}(x(t), y(t)), \frac{\partial f}{\partial y}(x(t), y(t)) \right) \cdot (a, b)$$

$$\frac{dz}{dt}(t=0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right) \cdot \vec{\ell}$$

SKALARNI PROIZVOD

DEF. IZVOD FUNKCIJE $z = f(x, y)$ U TAČKI $M_0(x_0, y_0)$ U PRAVCU VEKTORA $\vec{\ell}$, $|\vec{\ell}| = 1$ JE

$$\frac{\partial f}{\partial \vec{\ell}}(M_0) = \left(\frac{\partial f}{\partial x}(M_0), \frac{\partial f}{\partial y}(M_0) \right) \cdot \vec{\ell}.$$

VEKTOR ~~IZRAZ~~ $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$ SE NAZIVA <u>GRADIJENT</u>.

$$\operatorname{grad} f(M_0) = \left(\frac{\partial f}{\partial x}(M_0), \frac{\partial f}{\partial y}(M_0)\right) \quad \text{ILI}$$ (P7)

$$\nabla f(M_0) = \left(\frac{\partial f}{\partial x}(M_0), \frac{\partial f}{\partial y}(M_0)\right) \quad \text{ILI}$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \leftarrow \text{NABLA OPERATOR}$$

AKO JE $f(x,y,z)$, ONDA JE $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$.

$$\boxed{\frac{\partial f}{\partial \vec{e}} = \nabla f \cdot \vec{e}}$$

- NA OSNOVU DEFINICIJE SKALARNOG PROIZVODA, IMAMO:

$$\frac{\partial f}{\partial \vec{e}} = |\nabla f| \cdot \underbrace{|\vec{e}|}_{=1} \cdot \cos\sphericalangle(\nabla f, \vec{e}) = |\nabla f| \cdot \cos\sphericalangle(\nabla f, \vec{e})$$

AKO JE $\nabla f \parallel \vec{e}$, ONDA JE $\cos\sphericalangle(\nabla f, \vec{e}) = \pm 1$

SLEDI DA JE IZVOD U PRAVCU JE MAKSIMALAN ($|\nabla f|$) U PRAVCU POZITIVNOG GRADIJENTA, A MINIMALAN U PRAVCU NEGATIVNOG GRADIJENTA.

DAKLE, FUNK. $f(x,y)$ SE NAJBRŽE MENJA U PRAVCU GRADIJENTA.