On Topological Image Analysis

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2D Square Grid

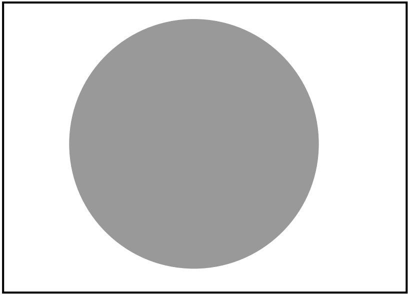
Ordered pairs $p = (i, j) \in \mathbb{Z}^2$, points or squares, with adjacency relation α defined based on coordinates.

- Adjacency α is symmetric.
- An α-path is a sequence of pixels, any two consecutive pixels are α-adjacent.
- A subset S of the grid is α-connected if there is an α-path in S connecting every two pixels of S.
- A connected component of S is a maximal connected subset.

A binary image is a rectangular subset of the grid, with black (object) or white (background) values assigned to pixels.
All notions generalize to 3D.

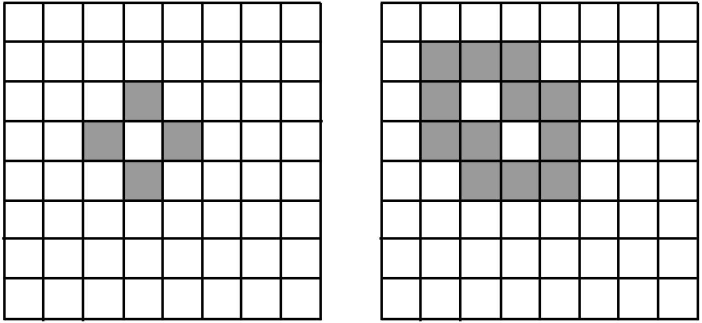
Topological "Problem"

Jordan curve theorem in $\mathbb{R}^2$: Every simple closed curve in the plane separates the plane in two disjoint open connected components. One of them is bounded, the other is unbounded.



Topological "Problem"

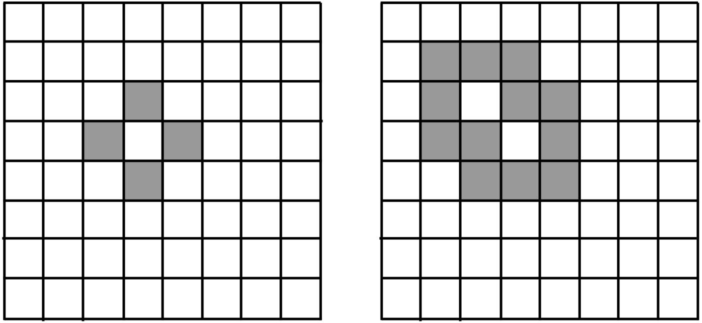
In the square grid, a simple closed α-curve is a finite α-connected set of pixels in which each pixel is α-adjacent to exactly two other pixels in the set.



Do these curves separate the digital plane if $\alpha = 0$ and if $\alpha = 1$? (α is the dimension of the intersection)

Solution

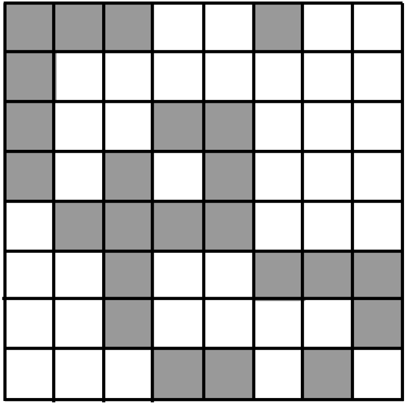
- A simple closed α-curve separates the grid in two disjoint $(1-\alpha)$-connected components.



Different adjacency α is used for object and background pixels.

Connected Component Labeling

Connectedness is an important topological property of a set.
What are the connected components?



Connected Component Labeling: Region-Growing

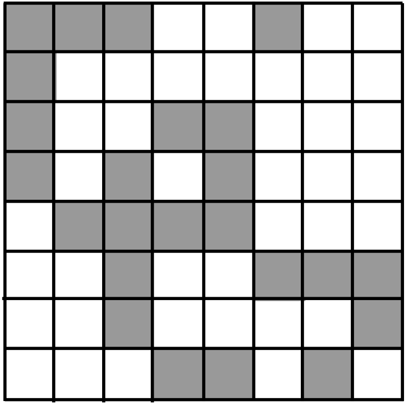
Scan image until a non-labeled black pixel p is found.

1. Label p with new label $L(p)$, put p in stack.
2. If the stack is empty, stop.
3. Pop r out of stack.
4. Label with $L(r)$ all non-labeled black pixels q α-adjacent to r, put them in stack.
5. Go to Step 2.

Repeat until all pixels are visited.

Exercise

In what order will the pixels be visited by the algorithm, if for each current pixel the order is clockwise starting from left?



Choose 0-adjacency for black pixels.

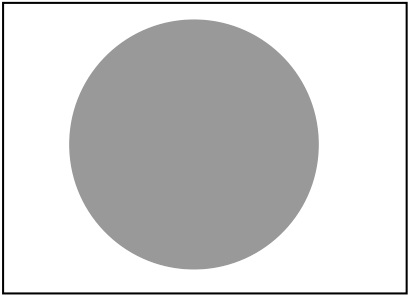
Connected Component Labeling: Rosenfeld-Pfaltz

- First scan:
  1. If the current black pixel p is adjacent to one or more black pixels labeled with the same label L, then label p with L.
  2. If p is adjacent to two or more black pixels labeled with different labels, then label p with the smallest of such labels L. Other labels are equivalent to L.
  3. Otherwise, assign a new label to p
- Determine equivalence classes of labels.
- Second scan:
  1. Replace every label with the representative of its equivalence class

Betti Numbers and Euler Characteristic

In 2D:

- β_0 - number of connected components
- β_1 - number of holes (connected components of the background, not counting the infinite background component)
- $\chi = \beta_0 - \beta_1$



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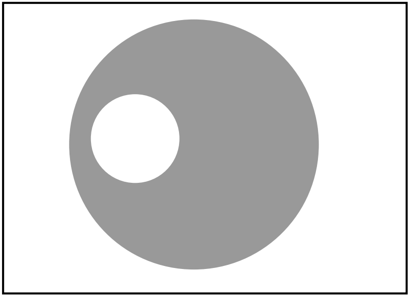

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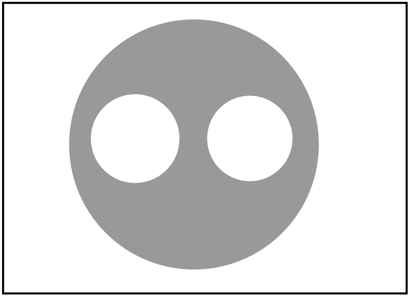

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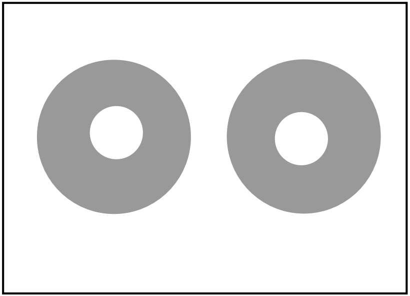

Betti Numbers and Euler Characteristic

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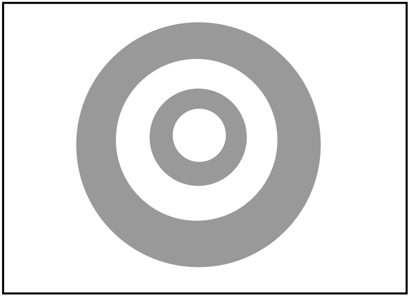


Betti Numbers and Euler Characteristic

In 3D:

- β_0 - number of connected components
- β_1 - number of tunnels (difficult to define and count)
- β_2 - number of voids (connected components of the background, excluding the infinite background component)
- $\chi = \beta_0 - \beta_1 + \beta_2$

An alternative view of the discrete grid

Consider also edges and vertices of squares (pixels) in the tessellation of the plane.

Cell complex $\Gamma = \bigcup_{k=0}^{m} \Gamma_k$

- finite set of cells (homeomorphic images of discs)
- cell interiors are disjoint
- boundary of each cell is a union of cells of lower dimension
- intersection of any two cells is composed of cells of lower dimension

Examples: Square and cubic grids, triangular and tetrahedral meshes.

For 0-adjacency

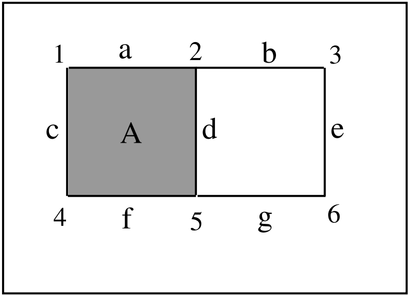
$\chi(\Gamma) = |\Gamma_0| - |\Gamma_1| + \ldots + (-1)^m |\Gamma_m|$

- $\chi(\Gamma) = v - e + f$ if $m = 2$
- $\chi(\Gamma) = v - e + f - c$ if $m = 3$.

Thus, β_1 can be easily computed in 3D as $\beta_0 + \beta_2 - \chi(\Gamma)$.

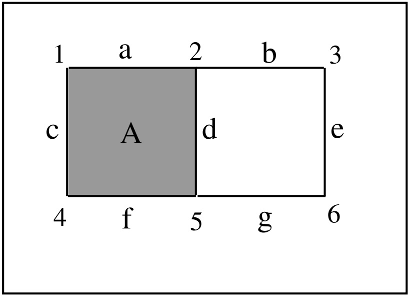
Example

Euler-Poincaré formula in 2D: $v - e + f = \beta_0 - \beta_1$.



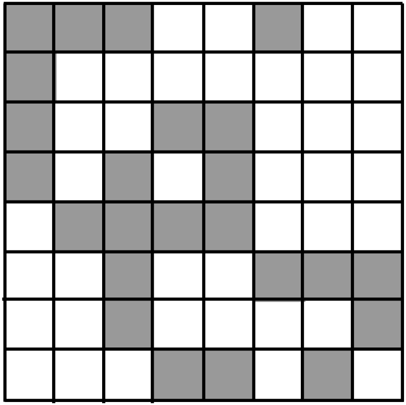
Example

Euler-Poincaré formula in 2D: $v - e + f = \beta_0 - \beta_1$.



$\beta_0 = 1,\ \beta_1 = 1 \qquad v = 6,\ e = 7,\ f = 1.$

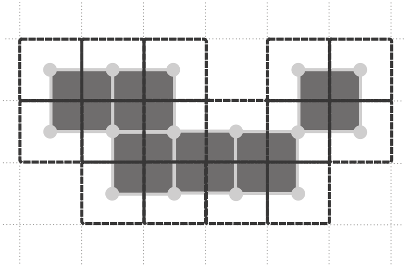
Counting the Cells?



Counting the Cells I

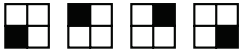
For each vertex (for each configuration of 2×2 pixels)

- count 1 vertex
- count $1/2$ of the edges
- count $1/4$ of the pixels



Counting the Cells I

Contributions of vertex configurations to the Euler characteristic.

type		Δv	Δe	Δf	$\Delta \chi$
Q_0		0	0	0	0
Q_1		1	1	1/4	1/4
Q_2		1	3/2	1/2	0
Q_3		1	2	3/4	-1/4
Q_4		1	2	1	0
Q_D		1	2	1/2	-1/2

For each vertex, all four incident pixels have to be visited.

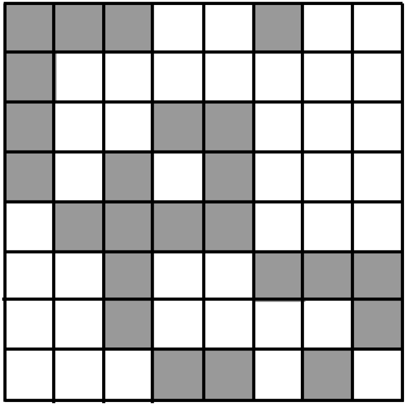
Counting the Cells II

Viewing from a fixed direction

- count components when they are created
- count holes when they are closed.

$\Delta\chi = 1$ $\quad$ $\Delta\chi = -1$ $\quad$ $\Delta\chi = -1$

Counting the Cells II



3D Objects

For manifold objects, we can count only the boundary cells

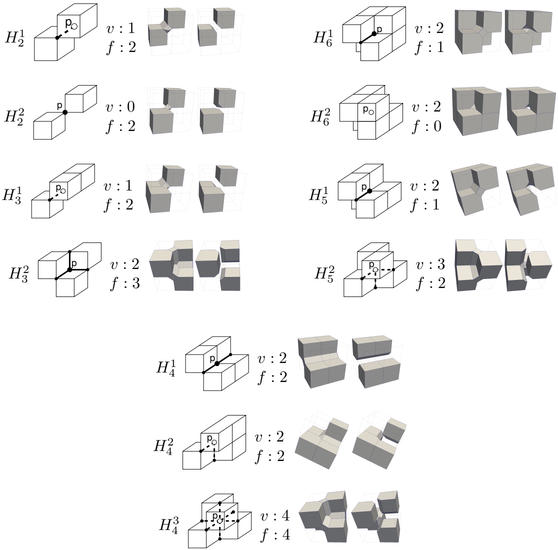
$$2\chi(O) = \chi(\partial O) = v^* - e^* + f^*$$

or use the discrete Gauss-Bonnet theorem

$$\chi(\partial O) = \frac{1}{2\pi} \sum_{v \in \partial O} \delta(v) = v^* - f^*$$

The angular deficit $\delta(v)$ is the amount by which the sum of the face angles at v differs from 2π.
For non-manifold objects, some vertices are counted twice.

3D Objects



H_2^1 p v : 1 f : 2
H_6^1 p v : 2 f : 1
H_2^2 p v : 0 f : 2
H_6^2 p v : 2 f : 0
H_3^1 p v : 1 f : 2
H_5^1 p v : 2 f : 1
H_3^2 p v : 2 f : 3
H_5^2 p v : 3 f : 2
H_4^1 p v : 2 f : 2
H_4^2 p v : 2 f : 2
H_4^3 p v : 4 f : 4

Homology Generators

Boundary matrix ∂_k:

- columns are labeled by k-cells
- rows are labeled by $(k-1)$-cells
- matrix element is 1 (0) if the $(k-1)$-cell is (not) in the boundary of the k-cell

Take ∂_k to the reduced form (1's at the beginning of the main diagonal, 0's elsewhere):

- exchange two rows (columns)
- add mod 2 one row (column) to another

Homology Generators

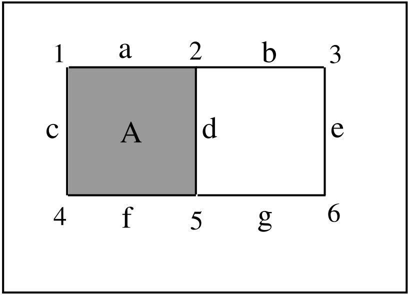
- number of zero columns = number of k-cycles z_k
- number of non-zero rows = number of $(k-1)$-boundaries b_{k-1}
- $\beta_k = z_k - b_k$

Do the same operations on row/column labels.

- labels of zero columns are cycles
- labels of non-zero rows are boundaries

Cycles that are not boundaries are homology generators.

Example



Thanks

Questions?