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Binary tomography on the isometric tessellation involving pixel shape orientation

Benedek Nagy[1], *Tibor Lukić*[2,*]

[1] *Faculty of Arts and Sciences, Eastern Mediterranean University, Famagusta, North Cyprus, Mersin-10, Turkey.*
[2] *Faculty of Technical Sciences, University of Novi Sad, 2100 Novi Sad, Serbia.*
* *E-mail: tibor@uns.ac.rs*

Abstract: In this paper, a tomography reconstruction problem of binary images is considered on the isometric grid. On this grid, the triangle pixels have two types of orientations, accordingly we call them delta or nabla shape pixels. The proposed reconstruction method uses data of projections of three natural directions. They are the lane directions of the triangular tessellation (these directions are somewhat analogous to row/column directions on the rectangular grids). The projection ray, penetrating through a grid lane, now not passing through the middle of pixels (i.e., through middle line of triangle shape pixels), as usually taken, but little bit shifted from the middle parallel to the lane. This method provides the exact information about the number of nabla and delta shape triangle pixels in each lane of the image. This additional information is included into the reconstruction process to improve the quality of reconstruction. We formulate the suggested model into an energy minimization problem and apply a gradient based approach for its minimization. We show and analyse various experimental results on test images. The presented approach shows both better quality reconstructions and shorter running time than the earlier approaches.

1 Introduction

Digital image processing works with images built up from a finite number of pixels. The square grid is the most wide spread and the most usually used. It is built up by square pixels, and the usual shape of the images is rectangle. Most of the image processing applications are based on algorithms that use the square grid; however, in the two dimensional space (i.e., in the plane), the isometric grid (that is built up by triangle pixels) and hexagonal grid are applicable alternatives [1]. Theory and applications of the hexagonal grid have been investigated around five decades ago, almost at the same time they have been started on the square grid, [2, 3] and [4]. In this paper, we are dealing with puzzles called tomography problems on the isometric grid. In the past few years, several binary tomography reconstruction methods for triangular gird were proposed. We recall both stochastic- [5, 6] and deterministic [7] approaches, but also the most advanced and recently proposed one, based on the dense projection approach (DPA) [8]. The DPA method, which provides the best results so far, requires increased projection emission/radiation, increased memory storage and thus, a larger amount of running time. For this reason, it is very important to investigate newer methods which provide at least as good result as the DPA, but with less radiation. In this paper, we propose a new, so called, shifted projection method. This approach gives the same quality of reconstruction results as the DPA, and more importantly, it does not need increased amount of projection data, that is, additional radiation of the considered object.

Further in this section, the general tomography model will be described and we are going to give a brief survey on triangular grid.

1.1 The tomography model on rectangular grid

In the following we give a short description of the considered tomography model based on the classical rectangular grid. The measured projection data are given by the projection vector $b \in \mathbb{R}^m$. The problem of reconstruction can be considered as a linear system of equations

$$Au = b, \quad A \in \mathbb{R}^{m\times n}, \quad b \in \mathbb{R}^m. \tag{1}$$

The matrix A is the projection matrix, whose each row corresponds to exactly one projection ray. The total number of projection rays is m. Each row entry $a_{i,\cdot}$ of A is determined to be equal with the length of the intersection of the projection ray passing through the pixel with the pixel itself, see Figure 1. The elements of vector b contain the measured m projection values, i.e., the sums of the products of the corresponding length of the projection ray through the pixel and its intensity. The image we want to reconstruct is denoted by the vector u. In the case of discrete tomography (DT) [9, 10] we assume that u contains pixels with finite/few number of given gray levels. A special and simplest case of DT is the binary tomography (BT), where the image to be reconstructed contains only 0's and 1's (background and object pixels), i.e., $u \in \{0,1\}^n$.

In the most of DT problems, the number of projection rays is significantly smaller than the number of image pixels, that is $m << n$. Therefore, the linear system (1) is underdetermined and, also of large scale, which means that the direct solution is not a valid option. We note that the problem of existence and uniqueness of the reconstruction from more than 2 projection directions is NP-complete, for further details see [11, 12]. Several successful energy-minimization type methods are proposed for rectangular grid, see for example [13–15], which motivate us to apply this approach in the case of triangular grid, discussed in this paper.

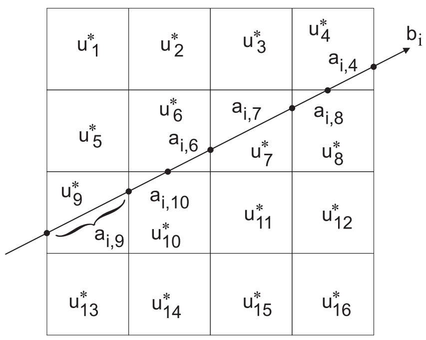

Fig. 1: Calculation of the projection value on a 4×4 image, where the ray goes through some image pixels: $b_i = a_{i,4}u^*_4 + a_{i,6}u^*_6 + a_{i,7}u^*_7 + a_{i,8}u^*_8 + a_{i,9}u^*_9 + a_{i,10}u^*_{10}$.

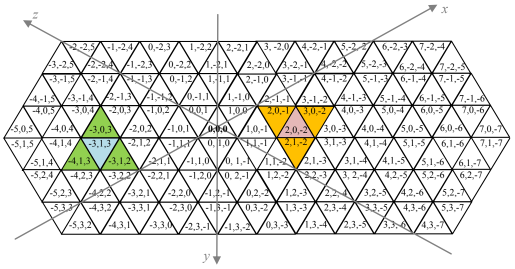

Fig. 2: The symmetric coordinate frame for the isometric tessellation, with a delta type (even) and a nabla type (odd) triangle and their side-neighbor triangles.

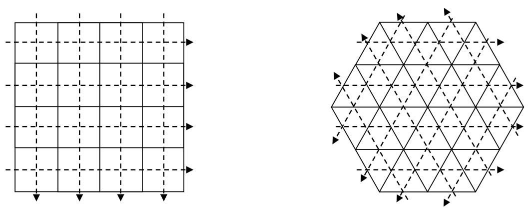

Fig. 3: The usual projection directions: rows and columns on the rectangular grid (left) and by three lane directions on the triangular grid (right).

1.2 The isometric grid

The Cartesian coordinate frame is a very efficient tool of the rectangular/square grid. Based on that this grid is the most widely used regular tessellation. The Cartesian coordinate frame gives its nice and efficient description. The columns and the rows of an image are addressed by two independent coordinates that are integers. The usual shape of the images is rectangle, as every shape has an embedding rectangle. The isometric (also called triangular) tessellation is also described in an elegant way by three coordinates, as it is presented in Figure 2. Even if the coordinates are dependent, the description keeps the symmetry of the tessellation. This tessellation is not a point lattice, since there are two different orientations of the pixels. The ones addressed by zero-sum triplets are called even pixels, they are △ (delta) pixels here. The others with triplets having sum 1 are the odd triangles; their shapes are ∇ (nabla). The coordinate triplets that address the triangles of the tessellation are exactly the integer triplets for which the sum of the three coordinates is in the set $\{0,1\}$. Various other properties of the coordinate frame and the isometric grid are presented in [16–18], including the neighbor realtions among the triangles. Each triangle has three side-neighbors, they are obtained by incresing/decreasing one of the coordinate values by 1 (their type is the opposite than the original triangle). The sets of pixels obtained by fixing one of the coordinate values are the lanes (e.g., $y = 3$ for the bottom lane in Figure 2). The natural directions of the tessellation are catched by lanes: their role is very similar to the roles of rows and columns in the rectangular tessellation. The natural shape of the images is hexagon on the triangular grid: every shape has an embedding hexagon [19, 20]. Thus, we consider binary images that have regular hexagonal shape.

We use projections only in the natural directions of the used tessellation. On the rectangular grid these directions are the usual horizontal and vertical directions (row and column sums, respectively). On the isometric tessellation the projection directions are provided by the lanes. In Figure 3 both the square and triangular grid can be seen having one projection for each column/row/lane. Discrete tomography with binary images has been investigated both on the rectangular tessellation based on these two projections (see [21, 22] for founding studies) and for the isometric tessellation by lane directions ([5, 7, 23]). Here, we continue to work on this topic based on an idea which gives additional information on the triangular grid.

In Section 2, the proposed shifted projection approach is presented, then Section 3 describes the used gradient based reconstruction algorithm, while in Section 4 the experimental evaluation of the new approach is shown. Finally, in Section 5, concluding remarks close the paper.

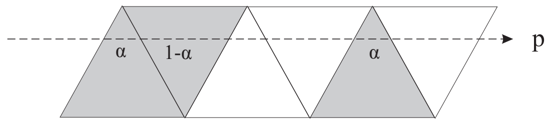

Fig. 4: A shifted projection ray goes inside a lane parallel to it.

2 Proposed shifted projection approach

We consider three natural directions, the lane directions, of projections of the triangular tessellation, these directions are analogous to the usual two directions used on the rectangular grid. In this paper, we investigate a projection approach when the projection ray goes through triangles parallel in lane directions, however we use the rays not along the middle of a lane, as it was naturally suggested earlier [5, 7]. The projection ray, that we use now, is not in the middle, see illustration in Figure 4. We use unit sidelength for the pixels of the grid. The value of α is the ray length through delta type triangles, while $1-\alpha$ is the length through pixels with nabla orientation. The value of α is from the interval $(0,1)$, where $\alpha \neq 0.5$. This "shifted" choice of α gives extra information for the reconstruction about the number of even and odd triangles forming the object to be reconstructed. In fact, the exact number of the object pixels of each type is determined for each lane. If α is an irrational number in $(0,1)$, it can easily be shown that the numbers of even and odd triangles are uniquely determined for each lane by the given projection value p, see Figure 4. Let s and l denote numbers of odd (∇) and even (△) pixels belonging to the object in a considered lane, respectively. The projection value p is calculated by

$$s \cdot (1-\alpha) \quad + \quad l \cdot \alpha \quad = \quad p. \tag{2}$$

Let us suppose that there are two different solutions of (2), $(s_1,l_1) \neq (s_2,l_2)$. It is easy to see, that in that case we come to the following equation

$$s_1 + \alpha(l_1 - s_1) = s_2 + \alpha(l_2 - s_2). \tag{3}$$

Because α is an irrational number, equation (3) implies that $s_1 = s_2$ and $l_1 = l_2$, i.e., the projection equation (2) has a unique solution.

As we can see from the above analysis, the proposed method based on shifted projections on the isometric tessellation provides some additional information about the triangle pixels of the object: the number of triangles of each orientation per lane. This information can significantly improve the reconstruction results. In contrast to the dense projection approach, proposed in [8], we achieve this extra information without duplicating the number of projection rays, i.e., without increasing the radiation dose in real applications.

In addition, we note that after the determination of the number of even and odd triangles per lane, by equation (2), we can easily calculate projection values corresponding to any other additional/virtual projection ray in the same lane, parallel with it. That is, if s and l are known, then p_2 is calculated by $s \cdot (1-\alpha_2) + l \cdot \alpha_2 = p_2$, for some $\alpha_2 \neq \alpha$. Therefore, taking one additional ray per lane, the same amount of projection data can be provided as in the case of the dense projection approach [8].

In practice, however, it is not necessary to use irrational numbers. Let α be chosen as a rational number $\frac{p}{q}$, where p and q are co-prime

IET Research Journals, pp. 1–6

integers. Knowing the size of the images, two different solutions implies that

$$s_1 - s_2 + \alpha((s_2 - s_1) + (l_1 - l_2)) = 0,$$

where from the fact that $s_1 - s_2$ is an integer number follows that $\alpha((s_2 - s_1) + (l_1 - l_2))$ is also an integer. If the maximal number of triangle pixels in any image lane is at most M, then we have that $|k| \leq M$, where $k = (s_2 - s_1) + (l_1 - l_2)$. Because $\frac{p}{q} \cdot k$ is an integer, we have that q must divides k. However, if we choose q to be greater than M, then it is not true and the uniqueness of the solution is ensured. We note, that in our experiments, when M is maximally 26, for α we can choose $\frac{22}{31}$.

3 Deterministic Reconstruction Method

Our method uses three projection directions that are orthogonal to the coordinate axes, i.e., our data is collected by lanes (Figure 2).

The binary tomography reconstruction problem can be described by applying a linear system of equations with binary constraints:

$$Au = b, \quad A \in \mathbb{R}^{m \times n}, \ u \in \{0,1\}^n, \, b \in \mathbb{R}^m. \tag{4}$$

Here, matrix A is the projection matrix. Each row of this matrix corresponds to one projection ray. The vector b contains the given data: coordinates of b represents the measured m projection values. The image to be reconstructed is the binary-vector u. Each row entry $a_{i,\cdot}$ of A is determined to be equal with the length of the intersection of the triangle pixel and the projection ray penetrating through it. According to the proposed shifted projection approach, see Section 2, the ray is not in the middle line of the triangles, which side is one. By α we denote the ray length through an even triangle, while $1-\alpha$ is the length through odd pixels, see Figure 4. In our experiments we set α to be equal to 0.71. The projection value given by a ray is computed similarly as it is described in the case of the square grid (see Figure 1). Three projection directions are used here, for each direction, projection for each lane is taken.

The binary tomography problem presented in the form of the linear system (4) is hard to be solved by algebraic reconstruction methods. Therefore, we reformulated it into an energy-minimization problem defined by

$$\min_{u \in \{0,1\}^n} E(u), \tag{5}$$

where the energy (or objective) function is given by

$$E(u) = \frac{1}{2}\left(\|Au - b\|_2^2 + \lambda \sum_i \sum_{j \in \Upsilon(i)} (u_i - u_j)^2 \right). \tag{6}$$

Here, in (6), $\|Au - b\|_2^2$ is the *data correlation* term, it measures the concurrence of a proposed solution u with the data given by the projection. The other part is the *smooth regularization* term; it is used to enforce the coherency of the solution. We apply this term based on the assumption that the image to be reconstructed is usually formed from relatively compact regions of triangles with homogeneous intensities. This assumption is true for most of real images; however, for randomly generated images it may not hold. The counterpart of this regularization on square grid is often used in discrete tomography methods [13, 14, 24], which additionally motivate our choice. By $\Upsilon(i)$ we denote the set of indices of the neighbor pixels of u_i. We defined it as follows. For a pixel (x,y,z) with $x+y+z=0$ ($\triangle$ shape) the triangles $(x+1,y,z)$ and $(x,y+1,z)$ are used; on the other hand, for odd pixels, (∇ shape), the triangle $(x,y,z-1)$ is used as their neighbors. The definition of $\Upsilon(i)$ ensures that each difference of two closest neighbor triangles is taken into account exactly once in the formula. The parameter $\lambda > 0$ is the balancing parameter between the above mentioned two terms, the data correlation and smoothing terms. In experimental part, we put its value to be 0.5.

Algorithm 1: SH-T.

Parameters: $\varepsilon_{out} > 0$; $\mu_\Delta > 0$.
$u^{init} = [0.5, 0.5, \ldots 0.5]^T$; $\mu = 0$; $u^{old} = u^{init}$;
while $\max_i \{\min\{u_i^{old}, 1 - u_i^{old}\}\} > \varepsilon_{out}$ (7)
do
  /* Choose u^{old} for the initial solution: */
  $u^{new} = \arg\left\{ \min_{u \in [0,1]^n} E_R(u;\mu) \right\}$; (8)
  $u^{old} = u^{new}$;
  $\mu = \mu + \mu_\Delta$; (9)
end

The problem (5) is reformulated in a way to have non-discrete and convex constraint set $[0,1]^n$:

$$\min_{u \in [0,1]^n} E_R(u;\mu) := E(u) + \frac{1}{2}\mu \langle u, \tau - u \rangle, \quad \mu > 0, \tag{10}$$

where $\langle u, \tau - u \rangle$ is a concave regularization term with $\tau = [1,1,\ldots,1]^T$. The task of this term is to ensure the final binary solution. Parameter μ regulates its influence. We note that the problem (10) is also known as *convex-concave* regularized problem, for more details see [25].

The energy function E_R is differentiable and therefore the problem (10) can be treated by a gradient based minimization algorithm. We use the Spectral Projected Gradient (SPG) iterative algorithm [26, 27] for this task. The reconstruction method, what we suggest, is presented in Alg. 1. The strategy is to solve a sequence of minimization problems (8), such that the next starts from the solution provided by the previous one. We note that this optimization strategy is first proposed and successfully applied in tomography reconstruction problem based on classical rectangular grid, see our earlier paper [13]. Now, we adapt this strategy for the case of triangular grid. In each step, the binarization parameter μ is increased by μ_Δ, see in (9). Based on empirical analysis, in our experiments it was chosen to be 0.05. The reconstruction process is terminated when the binary solution is accomplished. The parameter ε_{out} controls the fulfillment of this requirement in (7). We set its value as $\varepsilon_{out} = 10^{-3}$.

Although the pseudo code of the Alg. 1 in few lines resembles to similar algorithms on the rectangular grid [13–15], we underline again the main differences:

- in (8) computing the minimal energy uses the specific neighborhood Υ for the triangular grid (see also equation (6)),
- the projection value (b in equation (6)) is collected by three directions (according to the triangular grid), moreover the shifted projection approach is used.

4 Experimental evaluation

This section is dedicated to the experimental evaluation of the SH-T reconstruction algorithm, proposed in this paper. To the best of our knowledge, there are very few tomography methods on the triangular gird, see [5, 7, 8]. The DPA [8] method requires two projection rays per lane, which means twice as much amount of projection data ("projection dose") in comparison with the SH-T. The method which shows best performance, so far, and using the same, one projection ray per lane, approach is the deterministic SPG-T method, proposed and thoroughly evaluated in [7]. Therefore, experimental results obtained by SH-T are compared with the results obtained by SPG-T method. We have used MATLAB environment in both cases.

A set of 15 binary test images with regular hexagon shape with side length of 26 is used in the experiment. Each image contains 4056 triangles, and this resolution is comparable to the resolution of images 64x64 pixels in the rectangular grid. The test images are shown in Figure 5. These images were tested with both algorithms.

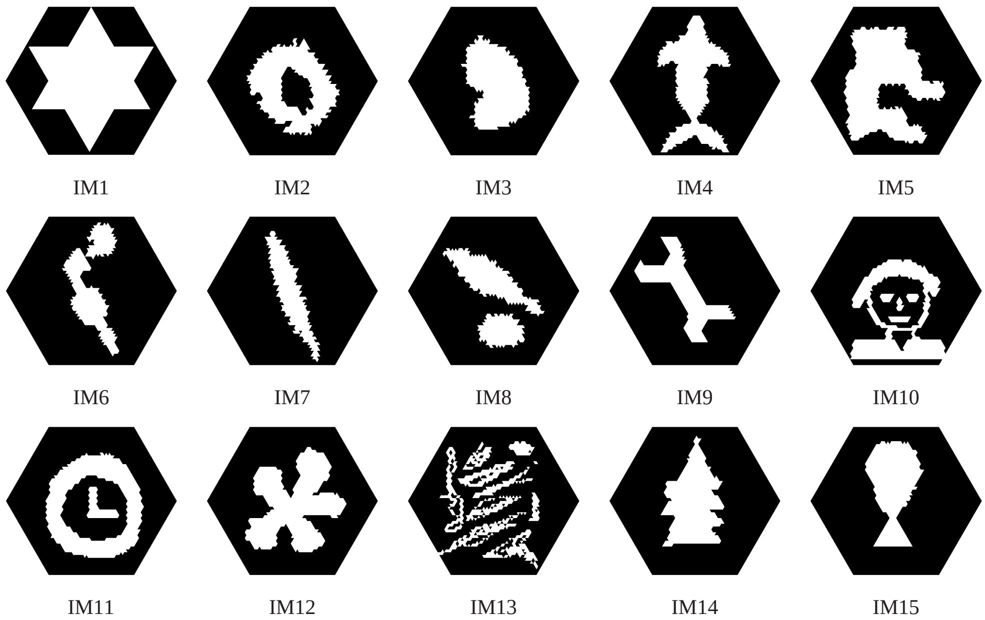

Fig. 5: Set of original test images. They all have regular hexagon shape and the same size of 26 by 26 by 26, i.e., 4056 pixels/triangles.

Table 1 Error measures (PE, $\nabla\Delta$ and PRE) for SPG-T and SH-T methods.

	SPG-T	SH-T	SPG-T	SH-T	SPG-T	SH-T	SPG-T	SH-T	SPG-T	SH-T
	IM1		IM2		IM3		IM4		IM5	
PE	0/0%	0/0%	90/2.21%	45/1.10%	32/0.78%	0/0%	67/1.65%	0/0%	42/1.03%	0/0%
$\nabla\Delta$	0	0	48	16	18	0	49	0	30	0
PRE	0	0	3.08	2.04	2.73	0	3.04	0	2.44	0
	IM6		IM7		IM8		IM9		IM10	
PE	55/1.35%	19/0.46%	63/1.55%	15/0.36%	92/2.26%	51/1.25%	5/0.12%	0/0%	293/7.22%	286/7.05%
$\nabla\Delta$	48	10	47	8	50	25	6	0	38	26
PRE	3.12	2.02	3.20	2.13	3.31	2.86	1.65	0	2.87	2.81
	IM11		IM12		IM13		IM14		IM15	
PE	451/11.11%	445/10.97%	4/0.9%	0/0%	963/23.74%	921/22.70%	5/0.12%	0/0%	4/0.09%	0/0%
$\nabla\Delta$	43	30	0	0	85	53	7	0	5	0
PRE	3.42	2.62	1.22	0	3.57	3.84	1.50	0	1.22	0

The quality of the obtained reconstructions has been measured by four error indicators. The pixel error, indicated by PE, shows the absolute number of misclassified triangles. It is calculated by $PE = \|u^r - u^{orig}\|_1$, where u^r and u^{orig} represents the image of the reconstruction result and the original image, respectively. Actually, it is identical to the Hamming distance between the two images. Its relative value, regarding to the image size, is calculated by $rPE = \frac{\|u^r - u^{orig}\|_1}{4056}$. The new information used in the SH-T method is the number of nabla and delta shape triangles per image lanes. Therefore, we also analyze the preservation of this information in reconstructions. The designation $\nabla\Delta$ shows the sum of the variation of the number of nabla and delta shape triangles in the reconstructed image from the exact number in each lane. It is calculated by

$$\nabla\Delta = \sum_{i=1}^{nr.lanes} |srec_i - sorig_i| + |lrec_i - lorig_i|,$$

where $srec_i$ and $lrec_i$ are the number of nabla and delta type triangles, respectively, in i-th lane of the reconstructed image, while $sorig_i$ and $lorig_i$ represents corresponding values extracted from the original image. It is clear that, during the reconstruction process, the given projection data b has to be preserved as much as possible. This aspect is expressed by the projection error PRE, which gives the matching of the reconstruction with the projection data. It is calculated as $\|Au^r - b\|_2$.

Several reconstructed images resulted by SPG-T and SH-T are shown in Figure 6. For every test image, below the reconstructed result, we show also the difference image presenting the positions of wrongly positioned triangles/pixels. Grey pixels indicates the missing ones, while white pixels shows those who are wrongly added. Both the pixel error PE and its relative value rPE (written in brackets), are given. As one can see, from the 15 reconstructions, in all cases the proposed SH-T method shows better results, according PE values, see Table 1. In most of the cases, this improvement is significant, because the PE values are reduced to less than their half in comparison with those obtained by SPG-T. Similarly, the presented projection errors PRE also give clear advantage to SH-T: those implicate that SH-T method provided better preservation of given projection data in all cases. Finally, considering $\nabla\Delta$ values in Table 1, we see that their are significantly reduced for SH-T method. This decrease is on average 66%. We find this important, because it shows that the information about numbers of ∇ and Δ shape pixels per lanes, obtained by the suggested shifted projection approach, contributes to better reconstruction results. In other

IET Research Journals, pp. 1–6

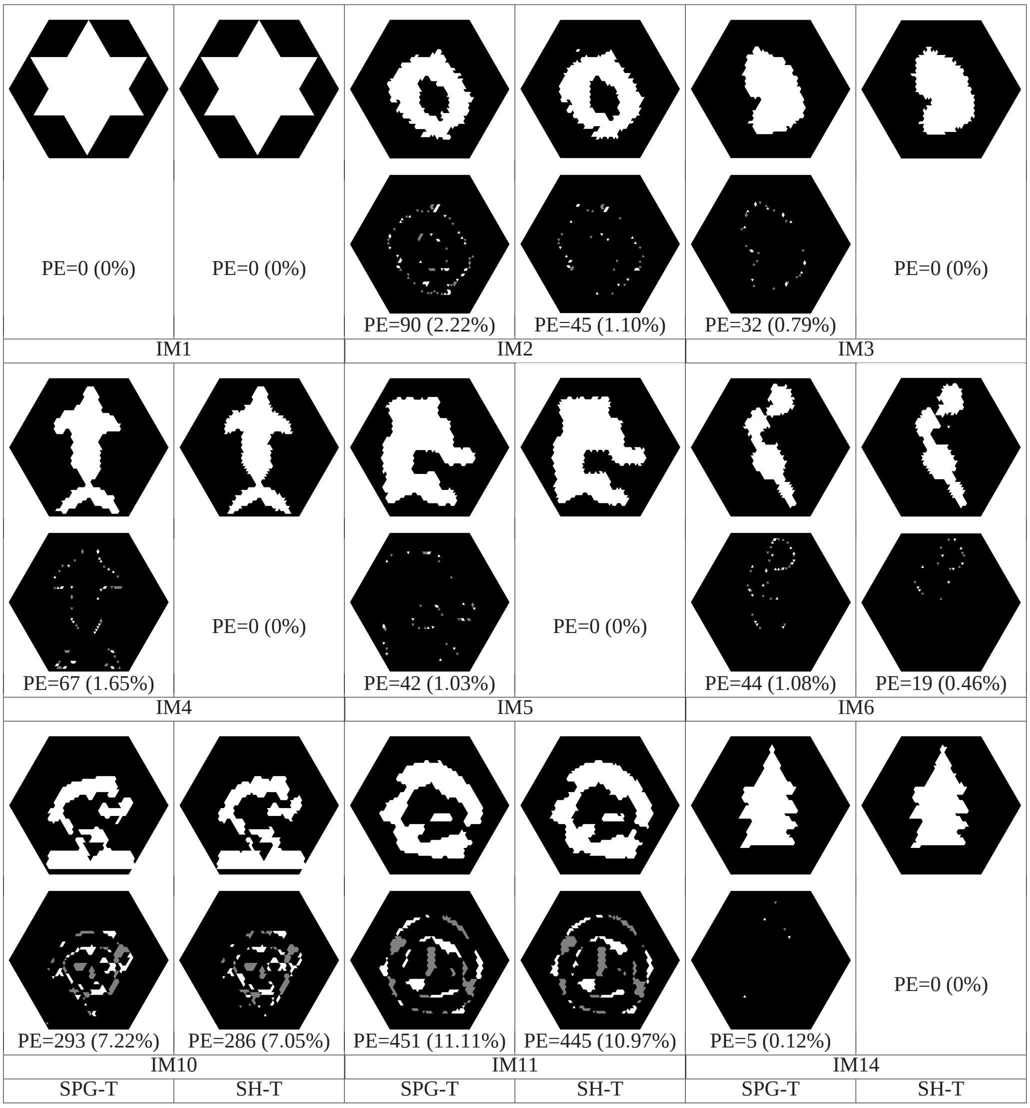

Fig. 6: Experimental results for several test images.

words, this experimentally confirms the correctness of the main idea suggested in this paper about utilization of the specificity of the triangular gird in tomography reconstructions.

5 Conclusions

Isometric or triangular tessellation allows the application of a new approach in binary tomography reconstruction. This paper points to this possibility. We considered the reconstruction which is based on three natural directions, namely lane directions of projections of the triangular tessellation – they are analogous to column and row directions of the rectangular grid. We used a novel approach: the projection ray, penetrating through the grid parallel with the lane, is shifted from the middle of triangular pixels. This approach allows that the numbers of even and odd triangles of the object are exactly computed in each lane. This additional information is included into the reconstruction process. We suggested a gradient based energy minimization reconstruction method. The energy function is formed as a sum of quadratic data fitting and smoothing terms. The applied energy minimization model and the optimization task is assigned to the convex-concave optimization framework based on Spectral Projected Gradient algorithm. Experiments on a set of 15 test images show advantage of the proposed method in comparison with an earlier suggested approach, in terms of quality of reconstructions.

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