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ABOUT SHAPE ORIENTATION AND ELONGATION

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Abstract. Shape orientation and shape elongation are well known shape descriptors and often used in different image processing applications. In this paper we will give a mathematical derivation of formulas for their measures calculation. Also, we will show that these formulas can be alternatively obtained by eigenvalues and eigenvectors of a corresponding matrix formed by geometrical moments.

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1. Introduction

Shape descriptors [1] belong to techniques which are often used in image analysis. Different objects appear on images and they should be classified depending on the task. Shape is an object characteristic. Shape descriptors determine shape characteristics or features which are suitable for numerical representation, usually, by mapping a certain shape characteristics to a set of numbers, for example to an interval $[0, 1]$. These numbers quantize the presence of the considered characteristic in shape. The most known shape descriptors include convexity, circularity, linearity, orientation, elongation, etc.

This paper deals with two shape descriptors: orientation and elongation. These two different descriptors can be derived by considering a common mathematical model. This model considers an integral of square distances of each shape point to a line. Determination of extreme values of this integral leads to the formulas for calculation of orientation and elongation. Another way to obtain these formulas is to determine the eigenvalues and eigenvectors of a specific matrix whose elements are geometrical moments. In this paper both ways of formula derivation will be considered and showed that the obtained results are same.

There are a number of recently published papers [2, 3, 4] where different aspects and modifications of orientation and elongation are analyzed. Application of these descriptors in image processing has a spread spectrum. We just mention the denoising problem [5, 6] and the tomography reconstruction reconstruction problem which is improved by shape orientation based regularization, see [7].

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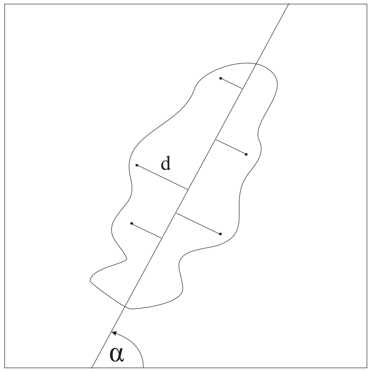


Figure 1: Illustration of a shape and its orientation represented by the angle α. The letter d indicates distance of object points to the line.

2. Shape orientation and elongation

Let $u : \Omega \longrightarrow [0,1]$, where $\Omega \subseteq \mathbb{R}^2$, be an image function representing a shape. Consider a line given by its normal equation

$$-\, x \cdot \sin\alpha \; + \; y \cdot \cos\alpha \; = \; \rho, \tag{2.1}$$

where $\alpha \in [0, \pi]$ is the line slope on the x-axis and $|\rho|$ is the orthogonal distance of the origin from the line, $-\infty < \rho < +\infty$. It is easy to see, that the square distance of a given point (x, y) to the line (2.1) is

$$d^2(x, y, \alpha, \rho) \; = \; (-x \cdot \sin\alpha + y \cdot \cos\alpha - \rho)^2.$$

Let consider the following integral

$$F(\alpha, \rho) = \iint_{\Omega} u(x, y) d^2(x, y, \alpha, \rho) dx dy, \tag{2.2}$$

where Ω is the image domain. This integral is proportional to the average square distance of shape points, represented by u, to the line determined by α and ρ. The *shape orientation* is defined as the slope of a line which minimizes the integral (2.2), i.e. the function $F(\alpha, \rho)$. This line is also called the axis of the least second moment of inertia, see [8]. The line which minimizes the integral $F(\alpha, \rho)$ must satisfy the equation $\frac{\partial F(\alpha,\rho)}{\partial \rho} = 0$, which is equivalent to $\iint_\Omega u(x, y)(x \sin\alpha - y \cos\alpha + \rho) dx dy = 0$, from what follows that

$$\begin{aligned} \rho &= -\sin\alpha \cdot \frac{\iint_\Omega x \cdot u(x, y) dx dy}{\iint_\Omega u(x, y) dx dy} + \cos\alpha \cdot \frac{\iint_\Omega y \cdot u(x, y) dx dy}{\iint_\Omega u(x, y) dx dy} \\ &= -x_c \cdot \sin\alpha + y_c \cdot \cos\alpha. \end{aligned}$$

The coordinates (x_c, y_c) mark the center of gravity or centroid of the shape. Hence, the minimizing line of (2.2) passes trough the shape centroid. If we insert $\rho = -x_c \cdot \sin\alpha + y_c \cdot \cos\alpha$ into $F(\alpha,\rho)$ we obtain

$$F(\alpha,\rho) = \iint_\Omega u(x,y)\left((x-x_c)\sin\alpha - (y-y_c)\cos\alpha\right)^2 dxdy = F(\alpha),$$

where we note that the integral depends only on α. $F(\alpha)$ can also be written in the following form

$$\begin{aligned}F(\alpha) = &\iint_\Omega u(x,y)\big((x-x_c)^2\sin^2\alpha - 2(x-x_c)(y-y_c)\sin\alpha\cos\alpha \\ &+(y-y_c)^2\cos^2\alpha\big)\,dxdy \\ = &\sin^2\alpha \iint_\Omega u(x,y)(x-x_c)^2\,dxdy + \cos^2\alpha \iint_\Omega u(x,y)(y-y_c)^2\,dxdy \\ &-\sin 2\alpha \iint_\Omega u(x,y)(x-x_c)(y-y_c)\,dxdy,\end{aligned}$$

that is, we obtain

$$F(\alpha) = \sin^2\alpha\ \overline{m}_{2,0}(u) + \cos^2\alpha\ \overline{m}_{0,2}(u) - \sin 2\alpha\ \overline{m}_{1,1}(u), \tag{2.3}$$

where $\overline{m}_{p,q}(u) = \iint_\Omega u(x,y)(x-x_c)^p(y-y_c)^q\,dxdy$ is a central moment of order $p+q$.

The angle α minimizing the function $F(\alpha)$ must satisfy the condition $\frac{dF(\alpha)}{d\alpha} = 0$, which gives the equation

$$\operatorname{tg} 2\alpha = \frac{\sin(2\alpha)}{\cos(2\alpha)} = \frac{2\cdot\bar{m}_{1,1}(u)}{\bar{m}_{2,0}(u) - \bar{m}_{0,2}(u)}. \tag{2.4}$$

We note, that the equation (2.4) gives two possible solutions. There are two angles (that are differ by 90°) where $F(\alpha)$ reaches its maximum and minimum. The minimum is taken for orientation. The orientation is an oriented angle, regarding the positive direction of abstracted x-axis, from interval $[0°, 180°]$, see Figure 1. Moments in equation (2.4) are translation invariant, therefore the orientation calculated by (2.4) is also translation invariant.

In the following, we determine the extremal values (maximum and minimum) of the function F. For that reason, we will insert the relation (2.4) in to the function (2.3). Applying trigonometric identities $\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$ and $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$, function $F(\alpha)$ transforms to the following form

$$\begin{aligned}F(\alpha) = &\frac{1}{2}(\bar{m}_{2,0}(u) + \bar{m}_{0,2}(u)) \\ &+\cos 2\alpha\left(\frac{1}{2}(\bar{m}_{0,2}(u) - \bar{m}_{2,0}(u)) - \frac{2\bar{m}_{1,1}^2(u)}{\bar{m}_{2,0}(u) - \bar{m}_{0,2}(u)}\right).\end{aligned} \tag{2.5}$$

Substituting the right side of the equation (2.4) into the trigonometric relation $\cos 2\alpha = \frac{1-\operatorname{tg}^2\alpha}{1+\operatorname{tg}^2\alpha} = \frac{1-\left(\frac{-1\pm\sqrt{1+\operatorname{tg}^2 2\alpha}}{\operatorname{tg} 2\alpha}\right)^2}{1+\left(\frac{-1\pm\sqrt{1+\operatorname{tg}^2 2\alpha}}{\operatorname{tg} 2\alpha}\right)^2}$, the following is obtained

$$\cos 2\alpha = \pm\frac{\bar{m}_{2,0}(u) - \bar{m}_{0,2}(u)}{\sqrt{(\bar{m}_{2,0}(u) - \bar{m}_{0,2}(u))^2 + 4\bar{m}_{1,1}^2(u)}}. \tag{2.6}$$

Inserting the relation into expression (2.3) we get two extremal values of F

$$\frac{1}{2}\left(\bar{m}_{2,0}(u)+\bar{m}_{0,2}(u)\pm\sqrt{(\bar{m}_{2,0}(u)-\bar{m}_{0,2}(u))^2+4\bar{m}_{1,1}^2(u)}\right),$$

more precisely, the maximum and minimum:

$$F_{max}=\tfrac{1}{2}\left(\bar{m}_{2,0}(u)+\bar{m}_{0,2}(u)+\sqrt{(\bar{m}_{2,0}(u)-\bar{m}_{0,2}(u))^2+4\bar{m}_{1,1}^2(u)}\right) \text{ and}$$

$$F_{min}=\tfrac{1}{2}\left(\bar{m}_{2,0}(u)+\bar{m}_{0,2}(u)-\sqrt{(\bar{m}_{2,0}(u)-\bar{m}_{0,2}(u))^2+4\bar{m}_{1,1}^2(u)}\right).$$

The standard *shape elongation* measure is defined as a ration between maximum and minimum values of the integral (2.2). This leads us to the following formula for calculation the shape elongation given by the image function u:

$$\mathcal{E}(u)=\frac{F_{max}}{F_{min}}$$

$$(2.7)\qquad =\frac{\bar{m}_{2,0}(u)+\bar{m}_{0,2}(u)+\sqrt{(\bar{m}_{2,0}(u)-\bar{m}_{0,2}(u))^2+4\bar{m}_{1,1}^2(u)}}{\bar{m}_{2,0}(u)+\bar{m}_{0,2}(u)-\sqrt{(\bar{m}_{2,0}(u)-\bar{m}_{0,2}(u))^2+4\bar{m}_{1,1}^2(u)}}.$$

This elongation measure gives values from interval $[1,+\infty)$. It reaches the minimal possible value of one for a circle, which is in accordance with the fact that circle has no elongation.

3. Orientation and elongation as eigenvectors and eigenvalues

The derived formulas for calculation of shape orientation (2.4) and elongation (2.7) can be derived alternatively considering eigenvectors and eigenvalues of the following matrix

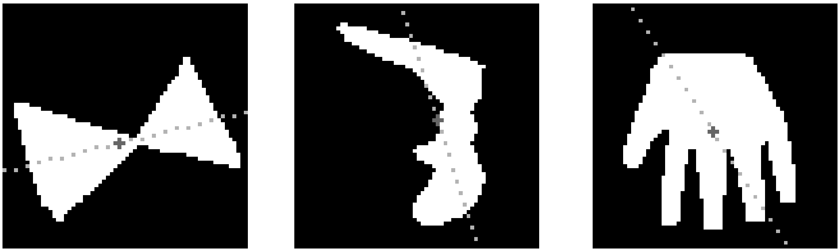
$$\begin{bmatrix}\bar{m}_{2,0}(u) & \bar{m}_{1,1}(u)\\ \bar{m}_{1,1}(u) & \bar{m}_{0,2}(u)\end{bmatrix}.$$

This matrix can be seen as a covariance matrix of two random variables, which represent x and y coordinates of shape points. In that case, the two eigenvectors give the first and second principal component of the shape points. The corresponding eigenvalues determine the magnitude of data variance in directions of eigenvectors. The largest eigenvalue is assigned to the first principal component. The direction of this component (or eigenvector) gives us the shape orientation. The ratio of largest and smallest eigenvalues determines the shape elongation. Let us show this by solving the following eigensystem

$$(3.1)\qquad \begin{bmatrix}\bar{m}_{2,0}(u) & \bar{m}_{1,1}(u)\\ \bar{m}_{1,1}(u) & \bar{m}_{0,2}(u)\end{bmatrix}\cdot\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\lambda\begin{bmatrix}v_1\\ v_2\end{bmatrix},$$

where $v_1,v_2\in\mathbb{R}$ are coordinates of an eigenvector, and $\lambda\in\mathbb{R}$ is an eigenvalue. Applying elementary calculus, we find that there are two eigenvalues

$$\lambda_{1,2}=\frac{1}{2}\left(\bar{m}_{2,0}(u)+\bar{m}_{0,2}(u)\pm\sqrt{(\bar{m}_{2,0}(u)-\bar{m}_{0,2}(u))^2+4\bar{m}_{1,1}^2(u)}\right).$$



$\alpha = 13.80°$ $\mathcal{E} = 6.38$ $\alpha = 107.74°$, $\mathcal{E} = 5.36$ $\alpha = 122.95°$ $\mathcal{E} = 1.33$

Figure 2: Binary phantom images. The dashed line and the cross, in each image, indicate the corresponding orientation axis and center of gravity, respectively. The angle α denotes the orientation in degrees and $\mathcal{E}$ the corresponding elongation value.

It is easy to see, that the elongation measure $\mathcal{E}(u)$ (2.7) can be obtained by $\frac{\lambda_1}{\lambda_2}$, which is what we wanted to show. Now, let us show that we can obtain the formula for orientation calculation (2.4). The largest eigenvector of (3.1), corresponds to λ_1, is

$$[v_1,\ v_2]^T = [\bar{m}_{2,0}(u) - \bar{m}_{0,2}(u) + \sqrt{(\bar{m}_{2,0}(u) - \bar{m}_{0,2}(u))^2 + 4\bar{m}_{1,1}^2(u)},\ 2\bar{m}_{1,1}(u)].$$

For the slope α of this vector regarding the x-axis we can write

$$\text{(3.2)} \qquad \operatorname{tg}\alpha = \frac{2\bar{m}_{1,1}(u)}{\bar{m}_{2,0}(u) - \bar{m}_{0,2}(u) + \sqrt{(\bar{m}_{2,0}(u) - \bar{m}_{0,2}(u))^2 + 4\bar{m}_{1,1}^2(u)}}.$$

Substituting $\operatorname{tg}\alpha$ from equation (3.2) into the well know trigonometric identity $\operatorname{tg}2\alpha = \frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}$, we get the equation which determines the shape orientation (2.4).

In the case when u is a digitized image, geometrical moments can be calculated by well-known digitized moments:

$$\bar{m}_{p,q}(u) = \sum_{(i,j)\in\Omega} u(i,j)\,(i - x_c)^p\,(j - y_c)^q\,,$$

where

$$(x_c, y_c) = \left(\frac{\sum_{(i,j)\in\Omega} u(i,j)\cdot i}{\sum_{(i,j)\in\Omega} u(i,j)},\ \frac{\sum_{(i,j)\in\Omega} u(i,j)\cdot j}{\sum_{(i,j)\in\Omega} u(i,j)} \right).$$

This allows application of the derived orientation (2.4) and elongation (2.7) formulas in many image processing problems. In Figure 2 we present 3 binary phantom images, presenting different shapes. We chose binary examples for simplicity, but formulas (2.4) and (2.7, of course, work well for grayscale images too.

4. Conclusion

In this paper two important shape descriptors are analyzed: orientation and elongation. A detailed mathematical derivation of formulas for their measures calculation is given using two different approaches: (1) by minimizing regarding a line the integral of the squared distances of the shape points to that line; (2) by determining eigenvalues and eigenvectors of matrix formed by corresponding geometrical moments. Authors hope that this short review paper will contribute to better understanding of the mathematical basis of considered shape descriptors.

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