MATEMATIKA 3

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Tema 1

Konvolucija

Laplasova transformacija i proizvod

U opštem slučaju,

$$\mathcal{L}\left\{f(t) \cdot g(t)\right\} \neq \mathcal{L}\left\{f(t)\right\} \cdot \mathcal{L}\left\{g(t)\right\}$$

$$\mathcal{L}\left\{(f \circ g)(t)\right\} \neq \mathcal{L}\left\{f(t)\right\} \circ \mathcal{L}\left\{g(t)\right\}$$

Primer

1. Neka je $f(t) = g(t) = 1$. Tada je

$$\mathcal{L}\{1\} \cdot \mathcal{L}\{1\} = \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}$$

$$\mathcal{L}\{1 \cdot 1\} = \frac{1}{s}$$

2. Neka je $f(t) = 1$ i $g(t) = t$. Tada je

$$F(s) = \mathcal{L}\{1\} = \frac{1}{s} \qquad G(s) = \mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{f(t)\} \circ \mathcal{L}\{g(t)\} = (F \circ G)(s) = F(G(s)) = F(\frac{1}{s^2}) = \frac{1}{\frac{1}{s^2}} = s^2$$

$$\mathcal{L}\{(f \circ g)(t)\} = \mathcal{L}\{f(g(t))\} = \mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \frac{1}{s^2}$$

Konvolucija

Definition

Neka su realne funkcije f i g takve da je $f(t) = g(t) = 0$ kada $t < 0$. Konvolucija funkcija $f(t)$ i $g(t)$ je funkcija definisana sa

$$\boxed{(f * g)(t) = \int_0^t f(u) \cdot g(t - u) du} \quad (1)$$

Laplasova transformacija konvolucije

Teorema

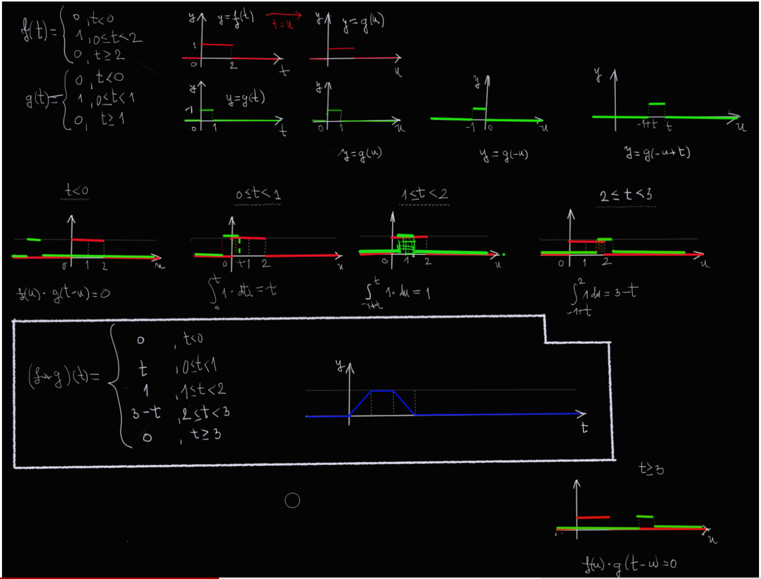
Neka je $\mathcal{L}\left\{f(t)\right\} = F(s)$ *i* $\mathcal{L}\left\{g(t)\right\} = G(s)$. *Tada je*

$$\boxed{\mathcal{L}\left\{(f * g)(t)\right\} = \mathcal{L}\left\{f(t)\right\} \cdot \mathcal{L}\left\{g(t)\right\}}$$

Posledica:

$$(f * g)(t) = \mathcal{L}^{-1}\{F(s) \cdot G(s)\}$$

Geometrijska interpretacija



Primer

Primer

Odrediti inverznu Laplasovu transformaciju funkcije $F(s) = \frac{1}{(s^2+a^2)^2}$.

Kako je $\frac{1}{(s^2+a^2)^2} = \frac{1}{s^2+a^2} \cdot \frac{1}{s^2+a^2}$ i $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin(at)$, dobijamo

$$
\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{a^2}\int\limits_0^t \sin(au)\sin(a(t-u))du \\
&= \frac{1}{a^2}\int\limits_0^t \sin(au)(\sin(at)\cos(au)-\cos(at)\sin(au))du \\
&= \frac{1}{a^2}\sin(at)\int\limits_0^t \frac{\sin(2au)}{2}du - \frac{1}{a^2}\cos(at)\int\limits_0^t \frac{1-\cos(2au)}{2}du \\
&= \frac{1}{a^2}\sin(at)\frac{\sin^2(at)}{2a} - \frac{1}{a^2}\cos(at)(\frac{1}{2}t - \frac{\sin(at)\cos(at)}{2a}) \\
&= \frac{1}{2a^3}(\sin^3(at) - at\cos(at) + \sin(at)\cos^2(at)) = \frac{1}{2a^3}(\sin(at) - at\cos(at))
\end{aligned}
$$

Tema 2

Heavisideov razvoj

Heavisideov razvoj

Teorema

Neka su $P = P(s)$ *i* $Q = Q(s)$ *polinomi stepena* m *i* n*, respektivno, za koje važi da je* $m < n$ *i* Q *ima* n *medjusobno različitih korena* $s_1, \dots, s_n$*. Tada je*

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \frac{P(s_1)}{Q'(s_1)}e^{s_1 t} + \frac{P(s_2)}{Q'(s_2)}e^{s_2 t} + \ldots + \frac{P(s_n)}{Q'(s_n)}e^{s_n t}.$$

Heavisideov razvoj

Neka je $Q(s) = a(s-s_1)\cdot\ldots\cdot(s-s_n)$ i $i \in \{1,\ldots,n\}$. Tada je

$$\frac{P(s)}{a(s-s_1)\cdot\ldots\cdot(s-s_n)} = \frac{a_1}{s-s_1} + \ldots + \frac{a_n}{s-s_n} \qquad \cdot (s-s_i)$$

$$\frac{P(s)(s-s_i)}{a(s-s_1)\cdot\ldots\cdot(s-s_n)} = \frac{a_1(s-s_i)}{s-s_1} + \ldots + \frac{a_1(s-s_i)}{s-s_i} + \ldots + \frac{a_n(s-s_i)}{s-s_n}$$

$$\frac{P(s)}{a(s-s_1)\ldots(s-s_{i-1})\cdot(s-s_{i+1})\ldots(s-s_n)} = \frac{a_1(s-s_i)}{s-s_1} + \ldots + a_1 + \ldots + \frac{a_n(s-s_i)}{s-s_n}$$

$\underline{s \to s_i}$

$$\begin{aligned} a_i &= \lim_{s\to s_i} \frac{P(s)}{a(s-s_1)\ldots(s-s_{i-1})\cdot(s-s_{i+1})\ldots(s-s_n)} \\ &= \frac{P(s_i)}{a(s-s_1)\ldots(s_i-s_{i-1})\cdot(s_i-s_{i+1})\ldots(s_i-s_n)} \\ &= \frac{P(s_i)}{Q'(s_i)}. \end{aligned}$$

Primer

Primer

Odrediti inverznu Laplasovu transformaciju funkcije

$$F(s) = \frac{2s+1}{s^3 - 6s^2 + 11s - 6}.$$

$$\frac{2s+1}{(s-1)(s-2)(s-3)} = \frac{a_1}{s-1} + \frac{a_2}{s-2} + \frac{a_3}{s-3}$$

$$a_1 = \lim_{s\to 1} \frac{2s+1}{(s-2)(s-3)} = \frac{3}{2} \quad a_2 = \lim_{s\to 2} \frac{2s+1}{(s-1)(s-3)} = -5 \quad a_3 = \lim_{s\to 3} \frac{2s+1}{(s-1)(s-2)} = \frac{7}{2}$$

$$\begin{aligned} F(s) &= \tfrac{3}{2}\tfrac{1}{s-1} - 5\tfrac{1}{s-2} + \tfrac{7}{2}\tfrac{1}{s-3} \\ f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \tfrac{3}{2}\mathcal{L}^{-1}\left\{\tfrac{1}{s-1}\right\} - 5\mathcal{L}^{-1}\left\{\tfrac{1}{s-2}\right\} + \tfrac{7}{2}\mathcal{L}^{-1}\left\{\tfrac{1}{s-3}\right\} \\ &= \tfrac{3}{2}e^{t} - 5e^{2t} + \tfrac{7}{2}e^{3t}. \end{aligned}$$

Tema 3

Primena na rešavanje običnih diferencijalnih jednačina

Obične diferecijalne jednačine sa konstatnim koeficijentima

Primer

Rešiti početni problem $y'' - 3y' + 2y = e^{2x}, \;\; y(0) = 0, \;\; y'(0) = 1.$

$$
\begin{aligned}
& \mathcal{L}\left\{y'' - 3y' + 2y\right\} = \mathcal{L}\left\{e^{2x}\right\} \\
\Leftrightarrow \quad & \mathcal{L}\left\{y''\right\} - 3\mathcal{L}\left\{y'\right\} + 2\mathcal{L}\left\{y\right\} = \frac{1}{s-2} \\
\Leftrightarrow \quad & s^2Y(s) - sy(0) - y'(0) - 3sY(s) + 3y(0) + 2Y(s) = \frac{1}{s-2} \\
\Leftrightarrow \quad & (s^2 - 3s + 2)Y(s) = \frac{1}{s-2} + 1 \\
\Leftrightarrow \quad & Y(s) = \frac{1}{(s-2)^2} \Leftrightarrow y(x) = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}.
\end{aligned}
$$

Ako je $f(x) = x$, onda je $F(s) = \frac{1}{s^2}$, odakle je $F(s-2) = \frac{1}{(s-2)^2}$. Tada je

$$y(x) = xe^{2x}$$

Obične diferecijalne jednačine sa konstatnim koeficijentima

$$\boxed{\mathcal{L}\left\{t^m y^{(n)}(t)\right\} = (-1)^m \left(\mathcal{L}\left\{y^{(n)}(t)\right\}\right)^{(m)}}$$

Primer

Primer

Naći rešenje diferencijalne jednačine

$$xy'' + (1 - 2x)y' - 2y = 0$$

za koje je $y(0) = 1, y'(0) = 0.$

Ako primenimo Laplasovu transformaciju i njene osobine na zadatu jednačinu, onda je

$$\begin{aligned}
& -(\mathcal{L}\{y''\})' + \mathcal{L}\{y'\} + 2(\mathcal{L}\{y'\})' - 2\mathcal{L}\{y\} = 0 \\
\Leftrightarrow \quad & -(s^2Y - sy(0) - y'(0))' + sY - y(0) + 2(sY - y(0))' - 2Y = 0 \\
\Leftrightarrow \quad & -(2sY + s^2Y' - 1) + sY - 1 + 2Y + 2sY' - 2Y = 0 \\
\Leftrightarrow \quad & (2s - s^2)Y' - sY = 0, \\
\Leftrightarrow \quad & \frac{dY}{Y} = -\frac{ds}{s-2} \Leftrightarrow Y = \frac{C}{s-2} \\
\Leftrightarrow \quad & y(x) = Ce^{2x} \text{ i } y(0) = 1 \\
\Leftrightarrow \quad & y(x) = e^{2x}.
\end{aligned}$$