MATEMATIKA 3

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Tema 1

Parametarske jednačine krivih u $\mathbb{R}^2$ i $\mathbb{R}^3$

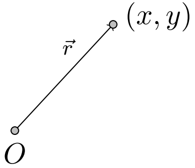
Vektor položaja tačke

Svakoj tački u prostoru $\mathbb{R}^2$ ili $\mathbb{R}^3$ možemo pridružiti njen vektor položaja $\vec{r}$. U prostoru $\mathbb{R}^2$:

$$\vec{r} = (x, y) = x\vec{\imath} + y\vec{\jmath}$$

U prostoru $\mathbb{R}^3$:

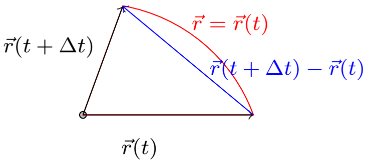
$$\vec{r} = (x, y, z) = x\vec{\imath} + y\vec{\jmath} + z\vec{k}.$$



Izvod vektorske funkcije

Neka je $\vec{r} : [a, b] \to \mathbb{R}^2$ ili $\vec{r} : [a, b] \to \mathbb{R}^3$ vektorska funkcija data sa:

$$\vec{r}(t) = (x(t), y(t)), t \in [a, b] \quad \text{tj.} \quad \vec{r}(t) = (x(t), y(t), z(t)), t \in [a, b].$$



Izvod vektorske funkcije $\vec{r}(t)$

Neka je $\vec{r} : [a, b] \to \mathbb{R}^3$ tj. $\vec{r} = \vec{r}(t)$ za $t \in [a, b]$.

$$\begin{aligned}
\Delta\vec{r}(t) &= \vec{r}(t+\Delta t) - \vec{r}(t) \\
&= (x(t+\Delta t), y(t+\Delta t), z(t+\Delta t)) - (x(t), y(t), z(t)) \\
&= (x(t+\Delta t) - x(t), y(t+\Delta t) - y(t), z(t+\Delta t) - z(t)) \\
\lim_{\Delta t\to 0} \frac{\Delta\vec{r}(t)}{\Delta t} &= \lim_{\Delta t\to 0} \frac{(x(t+\Delta t) - x(t), y(t+\Delta t) - y(t), z(t+\Delta t) - z(t))}{\Delta t} \\
&= \lim_{\Delta t\to 0} \left(\frac{x(t+\Delta t) - x(t)}{\Delta t}, \frac{y(t+\Delta t) - y(t)}{\Delta t}, \frac{z(t+\Delta t) - z(t)}{\Delta t} \right) \\
(\vec{r}(t))' &= \lim_{\Delta t\to 0} \frac{\Delta\vec{r}(t)}{\Delta t} = (\dot{x}(t), \dot{y}(t), \dot{z}(t)) \\
d\vec{r}(t) &= (\vec{r}(t))' dt = (\dot{x}(t)dt, \dot{y}(t)dt, \dot{z}(t)dt) = (dx(t), dy(t), dz(t)).
\end{aligned}$$

Brzina i ubrzanje

Geometrijski: $(\vec{r}(t))'$ je tangentni vektor na grafik vektorske funkcije $\vec{r} = \vec{r}(t)$ u tački t.

Brzina: Ako parametar t interpretiramo kao vreme: $(\vec{r}(t))'$daje promenu položaja tačke (pređeni put) u vremenu:

$$\boxed{\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = (\dot{x}(t), \dot{y}(t), \dot{z}(t)).}$$

Ubrzanje: Ubrzanje u nekoj tački predstavlja promenu brzine u vremenu:

$$\boxed{\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2} = (\ddot{x}(t), \ddot{y}(t), \ddot{z}(t)).}$$

Glatke krive

Neka je $i \in \{2, 3\}$.

Definicija

Za skup $L \subset \mathbb{R}^i$ kažemo da je prosta glatka kriva (ili gladak Žordanov luk) ako postoji interval $[a, b]$ i vektorska funkcija $\vec{r} : [a, b] \to \mathbb{R}^i$ za koje važi

(i) $L = \{\vec{r}(t) : t \in [a, b]\}$;

(ii) *$\vec{r}$ bijektivno preslikava skupa (a, b) na $L \setminus \{\vec{r}(a), \vec{r}(b)\}$;*

(iii) *$\vec{r}$ je neprekidno diferencijabilna funkcija na (a, b);*

(iv) $(\vec{r}(t))' \neq \vec{0}$ *za svako $t \in (a, b)$.*

Ako je $\vec{r}(a) = \vec{r}(b)$ kažemo da je kriva L zatvorena.

Kažemo da je tada sa

$$\boxed{\vec{r} = \vec{r}(t), \qquad t \in [a, b]}$$

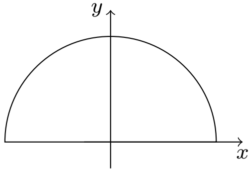
data glatka parametrizacija krive L.

Primer

Primer

Da li je $\vec{r}(t) = (\cos t, |\sin t|),\ t \in [0, \pi]$ *glatka parametrizacija neke krive?*

$$
\begin{aligned}
L &= \{(x, y) \in \mathbb{R}^2 : x = \cos t, y = |\sin t|, t \in [0, \pi]\} \\
&= \{(x, y) \in \mathbb{R}^2 : x = \cos t, y = \sin t, t \in [0, \pi]\} \\
&= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1, y \geq 0\}
\end{aligned}
$$



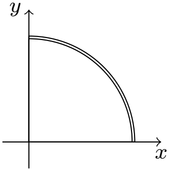
Da, data parametrizacija jeste glatka. Treba primetiti da je $|\sin t| = \sin t$, kada je $t \in [0, \pi]$.

Primer

Primer

Da li je $\vec{r}(t) = (|\cos t|, \sin t),\ t \in [0, \pi]$ *glatka parametrizacija neke krive?*

$$\begin{aligned} L &= \{(x,y) \in \mathbb{R}^2 : x = |\cos t|, y = \sin t, t \in [0,\pi]\} \\ &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1, x, y \geq 0\} \end{aligned}$$



Međutim, preslikavanje $\vec{r} : [0, \pi] \to L$ koje je definisano sa $\vec{r}(t) = (|\cos t|, \sin t)$ nije injektivno. Na primer

$$\vec{r}\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \vec{r}\left(\frac{3\pi}{4}\right).$$

Takvu parametrizaciju nećemo smatrati glatkom.

Primer

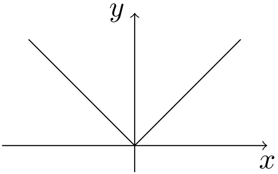
Primer

Da li je $\vec{r}(t) = (t, |t|),\ t \in [-2, 2]$ *glatka parametrizacija neke krive?*

Prvi izvod funkcije

$$y = y(t) = |t|,\ t \in [-2, 2]$$

nje definisan u tački $t = 0$.



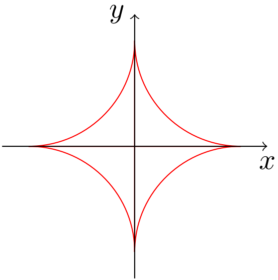
Primer

Primer

Da li je $\vec{r}(t) = (\cos^3 t, \sin^3 t), t \in [0, 2\pi]$ *glatka parametrizacija neke krive?*

$$\begin{aligned} \dot{x}(t) &= -3\cos^2 t \sin t \\ \dot{y}(t) &= 3\sin^2 t \cos t \end{aligned}$$

$$(\vec{r}(0))' = (\dot{x}(0), \dot{y}(0)) = (0, 0)$$



Po delovima glatke krive

Definicija

Skup $L \subset \mathbb{R}^3$ je po delovima glatka kriva ako je

$$L = L_1 \cup L_2 \cup \ldots \cup L_n,$$

gde su $L_1, L_2, \ldots, L_n$ glatke krive i svaki par L_i, L_j za $i \neq j$ može imati najviše konačno mnogo zajedničkih tačaka.

Putanja (ili prosta po delovima glatka kriva) je po delovima glatka kriva koja ne preseca samu sebe, osim eventualno u početnoj i kranjoj tački.

Primer

Primer

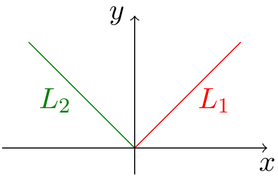
Skup tačaka

$$L = \{(x, y) \in \mathbb{R}^2 : x = t, y = |t|, t \in [-2, 2]\}.$$

predstavlja po delovima glatku krivu.

Skup L jednak je uniji skupova L_1 i L_2 gde je

$$\begin{aligned} L_1 &= \{(x, y) \in \mathbb{R}^2 : x = t, y = t, t \in [0, 2]\} \\ L_2 &= \{(x, y) \in \mathbb{R}^2 : x = t, y = -t, t \in [-2, 0]\} \end{aligned}$$



Primer

Primer

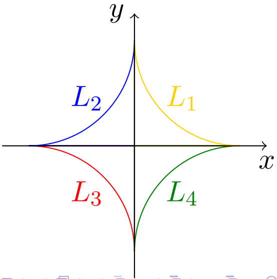
Skup tačaka

$$L = \{(x, y) \in \mathbb{R}^2 : x = \cos^3 t, y = \sin^3 t, t \in [0, 2\pi]\}.$$

predstavlja po delovima glatku krivu.

Za skup L važi $L = L_1 \cup L_2 \cup L_3 \cup L_4$, gde je

$$\begin{aligned}
L_1 &= \left\{(x, y) \in \mathbb{R}^2 : x = \cos^3 t, y = \sin^3 t, t \in \left[0, \tfrac{\pi}{2}\right]\right\}\\
L_2 &= \left\{(x, y) \in \mathbb{R}^2 : x = \cos^3 t, y = \sin^3 t, t \in \left[\tfrac{\pi}{2}, \pi\right]\right\}\\
L_3 &= \left\{(x, y) \in \mathbb{R}^2 : x = \cos^3 t, y = \sin^3 t, t \in \left[\pi, \tfrac{3\pi}{2}\right]\right\}\\
L_4 &= \left\{(x, y) \in \mathbb{R}^2 : x = \cos^3 t, y = \sin^3 t, t \in \left[\tfrac{3\pi}{2}, 2\pi\right]\right\}.
\end{aligned}$$



Tema 2

Primeri glatkih krivih u $\mathbb{R}^2$

Jednačine prave u $\mathbb{R}^2$

1. implicitni oblik: $Ax + By + C = 0$, $A \neq 0$ ili $B \neq 0$
2. eksplicitni oblik: $y = ax + b$
3. segmentni oblik: $\frac{x}{m} + \frac{y}{n} = 1$, $m, n \neq 0$
4. normalni oblik: $x \cdot \cos\varphi + y \cdot \sin\varphi = p, \quad \cos\varphi = \frac{p}{m}, \sin\varphi = \frac{p}{n}$

Jednačine prave u $\mathbb{R}^2$ - primer

1. eksplicitni oblik:
   $$\boxed{y = -2 \cdot x + 4}$$
2. implicitni oblik:
   $$\boxed{2x + 1 \cdot y - 4 = 0}$$
3. segmentni oblik:
   $$\boxed{\frac{x}{2} + \frac{y}{4} = 1}$$
4. normalni oblik:
   $$\boxed{\frac{2}{\sqrt{5}} \cdot x + \frac{4}{\sqrt{5}} \cdot y = \frac{1}{\sqrt{5}}}$$

Jednačine prave u $\mathbb{R}^2$ - primer

1. eksplicitni oblik:

   $$\boxed{y = -2 \cdot x + 4}$$

   $$y'(x) = \tan\alpha = -2$$

2. implicitni oblik:

   $$\boxed{2 \cdot x + 1 \cdot y - 4 = 0} \qquad \cdot\left(\frac{1}{4}\right) \qquad \cdot\left(\frac{1}{\sqrt{5}}\right)$$

   $$\vec{n} = (2, 1) = (F_x, F_y)$$

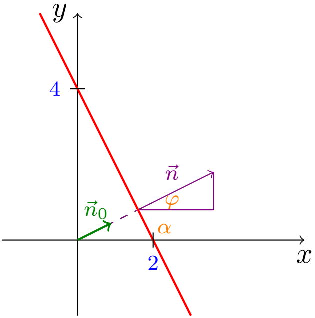
3. segmentni oblik:

   $$\boxed{\frac{x}{2} + \frac{y}{4} = 1}$$

4. normalni oblik:

   $$\boxed{\frac{2}{\sqrt{5}} \cdot x + \frac{1}{\sqrt{5}} \cdot y = \frac{4}{\sqrt{5}}}$$

   $$\vec{n}_0 = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$



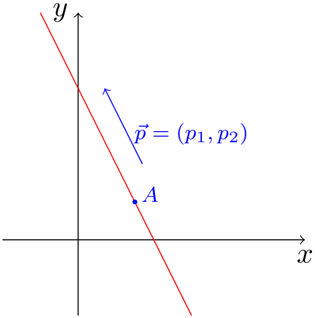
$$\vec{n} = \nabla F(x, y)$$

$$F(x, y) = 2x + y - 4$$

Jednačina prave kroz tačku u $\mathbb{R}^2$

Neka je data tačka $A(x_A, y_A)$ i vektor pravca prave $\vec{p} = (p_1, p_2)$.
Jednačina prave koja sadrži A i paralelna je sa $\vec{p}$ je sledećeg oblika:

$$p : \frac{y - y_A}{p_1} = \frac{x - x_A}{p_2}$$

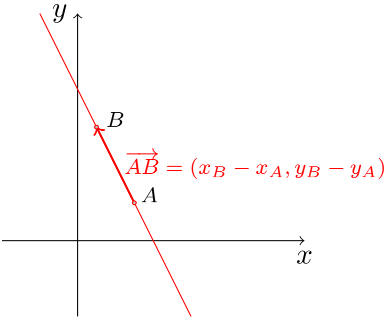


Jednačina prave kroz dve tačke u $\mathbb{R}^2$

Neka su $A(x_A, y_A)$ i $B(x_B, y_B)$ tačke prave p. Jedan vektor pravca prave p je:

$$\vec{p} = \overrightarrow{AB} = (x_B - x_A, y_B - y_A)$$

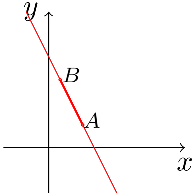
$$\frac{y - y_A}{\underbrace{y_B - y_A}_{p_2}} = \frac{x - x_A}{\underbrace{x_B - x_A}_{p_1}}$$



Parametrizacija duži i vektora u $\mathbb{R}^2$

Neka su date tačke $A(x_A, y_A)$ i $B(x_B, y_B)$.

$$
\begin{aligned}
\overline{AB} &= \{(x,y)\in\mathbb{R}^2 : \frac{y-y_A}{y_B-y_A} = \frac{x-x_A}{x_B-x_A} = t, t\in[0,1]\} \\
&= \{(x,y)\in\mathbb{R}^2 : x = (x_B - x_A)\cdot t + x_A, \\
&\qquad y = (y_B - y_A)\cdot t + y_A, t\in[0,1]\} \\
&= \{(x,y)\in\mathbb{R}^2 : x = t\cdot x_B + (1-t)\cdot x_A, \\
&\qquad y = t\cdot y_B + (1-t)\cdot y_A, t\in[0,1]\} \\
\overrightarrow{AB} &= \{(x,y)\in\mathbb{R}^2 : x = t\cdot x_B + (1-t)\cdot x_A, \\
&\qquad y = t\cdot y_B + (1-t)\cdot y_A, t\in[0,1]\} \\
\overrightarrow{BA} &= \{(x,y)\in\mathbb{R}^2 : x = t\cdot x_A + (1-t)\cdot x_B, \\
&\qquad y = t\cdot y_A + (1-t)\cdot y_B, t\in[0,1]\}
\end{aligned}
$$

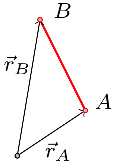


Parametrizacija duži i vektora u $\mathbb{R}^2$

U vektorskom obliku:

$$\vec{r} = \vec{r}_A + t \cdot (\vec{r}_B - \vec{r}_A)$$

$$\boxed{\vec{r} = t \cdot \vec{r}_B + (1 - t) \cdot \vec{r}_A, t \in [0, 1]}$$



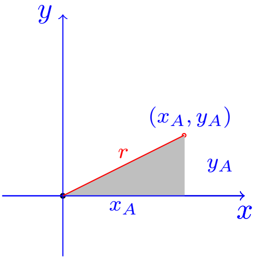
Krive drugog reda u $\mathbb{R}^2$

Nivo krive polinoma drugog stepena sa dve promenljive:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$$A \neq 0 \text{ ili } B \neq 0 \text{ ili } C \neq 0$$

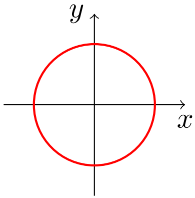
Veza između Dekartovih i polarnih koordinata



$$\cos\varphi = \frac{x_A}{r} \quad \Rightarrow \quad x_A = r\cos\varphi$$
$$\sin\varphi = \frac{y_A}{r} \quad \Rightarrow \quad y_A = r\sin\varphi$$

$\varphi \in [0, 2\pi)$
$r \geq 0$

Centralna kružnica u $\mathbb{R}^2$



Dekartove koordinate:

$$x^2 + y^2 = c^2 \quad (c > 0)$$

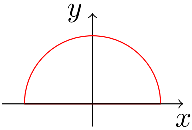
Polarne koordinate:

$$\left.\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \\ \varphi \in [0, 2\pi) \\ r \geq 0 \end{array}\right\} \Rightarrow (r\cos\varphi)^2 + (r\sin\varphi)^2 = c^2 \Rightarrow r = c \Rightarrow \begin{array}{l} x = c\cos\varphi \\ y = c\sin\varphi \\ \varphi \in [0, 2\pi) \end{array}$$

$$\boxed{L = \{(x, y) \in \mathbb{R}^2 : x = c\cos t, y = c\sin t, t \in [0, 2\pi)\}}$$

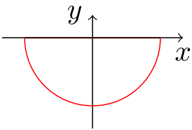
Deo centralne kružnice

$$
\begin{aligned}
L_1 &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = c^2, y \geq 0\} \\
&= \{(x,y) \in \mathbb{R}^2 : x = c\cos t, y = c\sin t, t \in [0,\pi]\} \\
&= \{(x,y) \in \mathbb{R}^2 : x = t, y = \sqrt{c^2 - t^2}, t \in [-c,c]\}.
\end{aligned}
$$



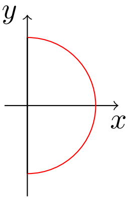
Deo centralne kružnice

$$
\begin{aligned}
L_2 &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = c^2, y \leq 0\} \\
&= \{(x,y) \in \mathbb{R}^2 : x = c\cos t, y = c\sin t, t \in [\pi, 2\pi]\} \\
&= \{(x,y) \in \mathbb{R}^2 : x = t, y = -\sqrt{c^2 - t^2}, t \in [-c, c]\}.
\end{aligned}
$$



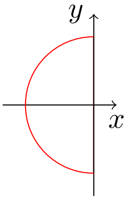
Deo centralne kružnice

$$
\begin{aligned}
L_3 &= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = c^2, x \geq 0\} \\
&= \{(x,y) \in \mathbb{R}^2 : x = c\cos t, y = c\sin t, t \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]\} \\
&= \{(x,y) \in \mathbb{R}^2 : x = \sqrt{c^2 - t^2}, y = t, t \in [-c, c]\}.
\end{aligned}
$$



Deo centralne kružnice

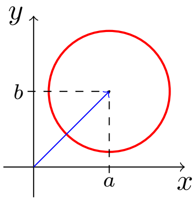
$$
\begin{aligned}
L_4 &= \{(x,y)\in\mathbb{R}^2 : \quad x^2+y^2=c^2, x\le 0\} \\
&= \{(x,y)\in\mathbb{R}^2 : \quad x = c\cos t, \\
&\qquad\qquad\qquad\qquad\quad y = c\sin t, \\
&\qquad\qquad\qquad\qquad\quad t\in[\tfrac{\pi}{2},\tfrac{3\pi}{2}]\} \\
&= \{(x,y)\in\mathbb{R}^2 : \quad x = -\sqrt{c^2-t^2}, \\
&\qquad\qquad\qquad\qquad\quad y = t, \\
&\qquad\qquad\qquad\qquad\quad t\in[-c,c]\}.
\end{aligned}
$$



Kružnica u $\mathbb{R}^2$

Dekartove koordinate:

$$(x-a)^2 + (y-b)^2 = c^2 \quad (c > 0)$$



Polarne koordinate i translacija:

$$\left.\begin{array}{l} x - a = r\cos\varphi \\ y - b = r\sin\varphi \\ \varphi \in [0, 2\pi) \\ r \geq 0 \end{array}\right\} \Rightarrow (r\cos\varphi)^2 + (r\sin\varphi)^2 = c^2 \Rightarrow r = c$$

$$\boxed{L = \{(x,y) \in \mathbb{R}^2 : x = c\cos t + a, y = c\sin t + b, t \in [0, 2\pi)\}}$$

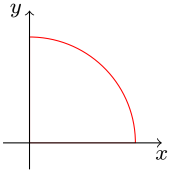
Primer

Primer

Napisati jednu glatku parametrizaciju krive

$$L = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1, x, y \geq 0\}.$$

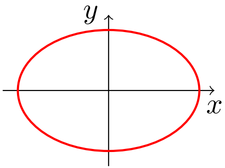
$$L = \{(x, y) \in \mathbb{R}^2 : x = \cos t, y = \sin t, t \in \left[0, \frac{\pi}{2}\right]\}$$



Centralna elipsa u $\mathbb{R}^2$

Dekartove koordinate:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a, b > 0)$$

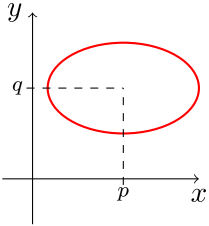


Uopštene polarne koordinate:

$$\begin{array}{l} \frac{x}{a} = r\cos\varphi \\ \frac{y}{b} = r\sin\varphi \\ \varphi \in [0, 2\pi) \\ r \geq 0 \end{array} \quad \Rightarrow \quad r = 1 \quad \Rightarrow \quad \begin{array}{l} x = a\cos\varphi \\ y = b\sin\varphi \\ \varphi \in [0, 2\pi) \end{array}$$

$$\boxed{L = \{(x, y) \in \mathbb{R}^2 : x = a\cos t, y = b\sin t, t \in [0, 2\pi)\}}$$

Elipsa u $\mathbb{R}^2$



Dekartove koordinate:

$$\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1 \quad (a, b > 0)$$

Uopštene polarne koordinate i translacija:

$$\begin{aligned} \frac{x-p}{a} &= r\cos\varphi \\ \frac{y-q}{b} &= r\sin\varphi \\ \varphi &\in [0, 2\pi) \\ r &\geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x - p &= a\cos\varphi \\ y - q &= b\sin\varphi \\ \varphi &\in [0, 2\pi) \end{aligned}$$

$$\boxed{L = \{(x, y) \in \mathbb{R}^2 : x = a\cos t + p, y = b\sin t + q, t \in [0, 2\pi)\}}$$