MATEMATIKA 3

Ivan Prokić
kabinet 117, F blok
prokic@uns.ac.rs
http://imft.ftn.uns.ac.rs/∼iprokic/

Andrea Karalić
kabinet 215, F blok
andrea.karalic@gmail.com

Novi Sad

Slajdove pripremila prof. dr Jovanka Pantović

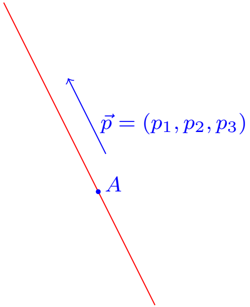
Tema 1

Primeri glatkih krivih u $\mathbb{R}^3$

Jednačina prave kroz tačku u $\mathbb{R}^3$

Neka je data tačka $A(x_A, y_A, z_A)$ i vektor pravca prave $\vec{p} = (p_1, p_2, p_3)$. Jednačina prave koja sadrži A i paralelna je sa $\vec{p}$ je sledećeg oblika:

$$p: \frac{x - x_A}{p_1} = \frac{y - y_A}{p_2} = \frac{z - z_A}{p_3}$$

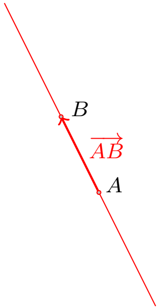


Jednačina prave kroz dve tačke u $\mathbb{R}^3$

Neka su $A(x_A, y_A, z_A)$ i $B(x_B, y_B, z_B)$ tačke prave p. Jedan vektor pravca prave p je:

$$\vec{p} = \overrightarrow{AB} = (x_B - x_A, y_B - y_A, z_B - z_A)$$

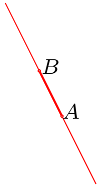
$$\frac{x - x_A}{\underbrace{x_B - x_A}_{p_1}} = \frac{y - y_A}{\underbrace{y_B - y_A}_{p_2}} = \frac{z - z_A}{\underbrace{z_B - z_A}_{p_3}}$$



Parametrizacija duži i vektora u $\mathbb{R}^3$

Neka su date tačke $A(x_A, y_A, z_A)$ i $B(y_A, y_B, z_B)$. Tada je

$$\begin{aligned}
\overline{AB} &= \{(x,y,z) \in \mathbb{R}^3 : x = t \cdot x_B + (1-t) \cdot x_A, \\
&\qquad y = t \cdot y_B + (1-t) \cdot y_A, \\
&\qquad z = t \cdot z_B + (1-t) \cdot z_A, t \in [0,1]\} \\
\overrightarrow{AB} &= \{(x,y,z) \in \mathbb{R}^3 : x = t \cdot x_B + (1-t) \cdot x_A, \\
&\qquad y = t \cdot y_B + (1-t) \cdot y_A, \\
&\qquad z = t \cdot z_B + (1-t) \cdot z_A, t \in [0,1]\} \\
\overrightarrow{BA} &= \{(x,y,z) \in \mathbb{R}^3 : x = t \cdot x_A + (1-t) \cdot x_B, \\
&\qquad y = t \cdot y_A + (1-t) \cdot y_B, \\
&\qquad z = t \cdot z_A + (1-t) \cdot z_B, t \in [0,1]\}
\end{aligned}$$

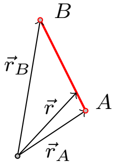


Parametrizacija duži i vektora u $\mathbb{R}^3$

U vektorskom obliku:

$\vec{r} = \vec{r}_A + t \cdot (\vec{r}_B - \vec{r}_A)$

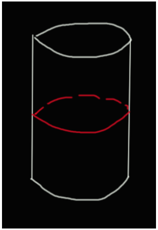
$$\boxed{\vec{r} = t \cdot \vec{r}_B + (1 - t) \cdot \vec{r}_A, t \in [0, 1]}$$



Presek ravni i kružnog cilindra

Neka je $c \in \mathbb{R}^+$.

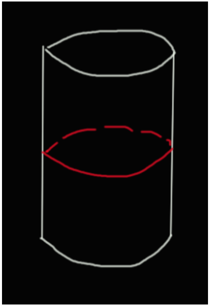
$$
\begin{aligned}
L &= \{(x,y,z) \in \mathbb{R}^2 : \quad x^2 + y^2 = c^2, z = z_1\} \\
&= \{(x,y,z) \in \mathbb{R}^2 : \quad x = c\cos t, \\
&\qquad\qquad\qquad\qquad\qquad y = c\sin t, \\
&\qquad\qquad\qquad\qquad\qquad z = z_1, \\
&\qquad\qquad\qquad\qquad\qquad t \in [0, 2\pi]\}.
\end{aligned}
$$



Presek ravni i kružnog cilindra

$$L = \{(x,y,z) \in \mathbb{R}^2 : \quad x^2 + y^2 = 9, x + y + z = 5\}$$

$$= \{(x,y,z) \in \mathbb{R}^2 : \quad \begin{array}{l} x = 3\cos t, \\ y = 3\sin t, \\ z = 5 - 3\cos t - 3\sin t, \\ \\ t \in [0, 2\pi]\}. \end{array}$$



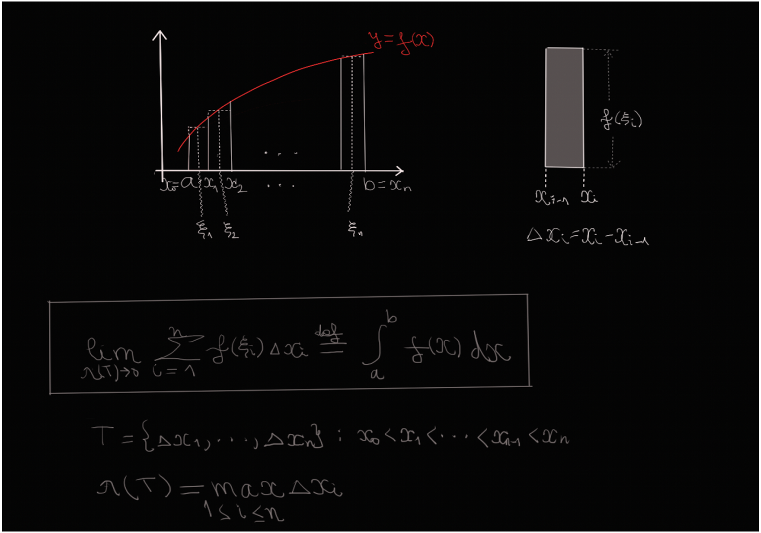
Cilindrična zavojnica

$$L = \{(x,y,z) \in \mathbb{R}^2 : \begin{aligned} & x = 3\cos t, \\ & y = 3\sin t, \\ & z = t, \\ & t \in [0, 2\pi]\}. \end{aligned}$$

Tema 2

Dvostruki integral

Određeni integral



Dvostruki integral

Neka je funkcija $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^2$, ograničena nad D, gde je D ograničena i zatvorena oblast i neka je d rastojanje na $\mathbb{R}^2$.
Podela:

$$T = \{D_1, D_2, \ldots, D_n\}$$

(i) $\emptyset \neq D_i \subseteq D$ za svako $i \in \{1, \ldots, n\}$,

(ii) $D = \bigcup\limits_{i=1}^{n} D_i$ i

(iii) $i \neq j \Rightarrow D_i^o \cap D_j^o = D$, za sve $i, j \in \{1, \ldots, n\}$.

Izaberimo za svako $i \in \{1, \ldots, n\}$ (proizvoljno) tačku

$$M_i \in D_i.$$

$$\lambda(T) = \max_{1 \leq i \leq n} d(D_i) \qquad d(D_i) = \max_{x,y \in D_i} d(x, y)$$

Dvostruki integral

Integralna suma:

$$s(f,T) = \sum_{i=1}^{n} f(M_i)\Delta D_i$$

(funkcije f po podeli T i izboru tačaka $M_i \in D_i$, $i = 1, 2, \ldots, n$)
Dvostruki integral:

$$\boxed{\iint\limits_D f(x,y)dxdy = \lim_{\substack{\lambda(T) \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(M_i)\Delta D_i}$$

Primer - po definiciji

Primer

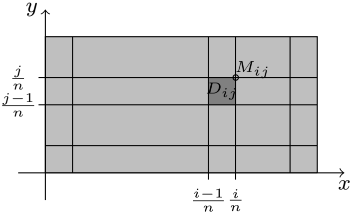
Izračunati po definiciji integral $I = \iint\limits_D x^2y^2dxdy.$

$$D = \{(x,y) \in \mathbb{R}^2 : \ 0 \leq x \leq 2, \ 0 \leq y \leq 1\}.$$

Izvršićemo podelu oblasti D na sledeći način:

$$D_{ij} = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$$

gde je $i = 1, \ldots, 2n, \ j = 1, \ldots, n.$



Površine posmatranih pravougaonika D_{ij} i parametar podele su

$$\Delta D_{ij} = \frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \qquad \lambda(T) = \frac{\sqrt{2}}{n}.$$

Primer - po definiciji

Ako $n \to \infty$, onda $\lambda(T) \to 0$. Izaberimo u svakom kvadratu D_{ij} tačku

$$M_{ij}\left(\frac{i}{n}, \frac{j}{n}\right) \quad i \in \{1, \ldots, 2n\} \quad j \in \{1, \ldots, n\}.$$

Traženi dvostruki integral je

$$\begin{aligned} I &= \iint_D x^2y^2dxdy = \lim_{n\to\infty} \sum_{i=1}^{2n}\sum_{j=1}^{n} f(M_{ij})\Delta D_{ij} \\ &= \lim_{n\to\infty} \sum_{i=1}^{2n}\sum_{j=1}^{n} \frac{i^2}{n^2}\frac{j^2}{n^2}\frac{1}{n^2} \\ &= \lim_{n\to\infty} \frac{1}{n^6}\sum_{i=1}^{2n} i^2 \sum_{j=1}^{n} j^2 = \lim_{n\to\infty} \frac{1}{n^6}\frac{2n(2n+1)(4n+1)}{6}\frac{n(n+1)(2n+1)}{6} = \frac{8}{9}. \end{aligned}$$

Osobine dvostrukog integrala

Neka su $\alpha, \beta \in \mathbb{R}$ i pretpostavimo da postoje integrali $\iint\limits_{D} f(x,y)dxdy$, $\iint\limits_{D_1} f(x,y)dxdy$, $\iint\limits_{D_2} f(x,y)dxdy$ i $\iint\limits_{D} g(x,y)dxdy$. Tada važe sledeće osobine:

(1) $\iint\limits_{D} \big(\alpha f(x,y) + \beta g(x,y)\big)dxdy = \alpha \iint\limits_{D} f(x,y)dxdy + \beta \iint\limits_{D} g(x,y)dxdy$,

(2) $\iint\limits_{D} f(x,y)dxdy = \iint\limits_{D_1} f(x,y)dxdy + \iint\limits_{D_2} f(x,y)dxdy$,
gde je $D = D_1 \cup D_2$ i $D_1^o \cap D_2^o = \emptyset$.

Osobine dvostrukog integrala

(3) Površina ΔD figure $D \subset \mathbb{R}^2$ data je integralom

$$\boxed{\Delta D = \iint\limits_D dxdy.}$$

(4) Neka je $f(x,y) \geq 0$, $(x,y) \in D \subseteq \mathbb{R}^2$. Zapremina ΔV oblasti

$$V = \{(x,y,z) \in \mathbb{R}^3 : \ (x,y) \in D \subset \mathbb{R}^2, \ 0 \leq z \leq f(x,y)\}$$

data je sa

$$\boxed{\Delta V = \iint\limits_D f(x,y)dxdy.}$$

Neka je $f(x,y) \leq 0$, $(x,y) \in D \subseteq \mathbb{R}^2$. Ako je oblast

$$V = \{(x,y,z) \in \mathbb{R}^3 : \ (x,y) \in D \subset \mathbb{R}^2, \ f(x,y) \leq z \leq 0\}$$

onda je zapremina

$$\Delta V = -\iint\limits_D f(x,y)dxdy.$$

Izračunavanje - pravougaona oblast

Teorema

Ako je f integrabilna funkcija i

$$D = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d\}.$$

onda je

$$\begin{aligned} \iint\limits_D f(x,y)dxdy &= \int\limits_a^b \left(\int\limits_c^d f(x,y)dy \right) dx = \int\limits_a^b dx \int\limits_c^d f(x,y)dy \\ \iint\limits_D f(x,y)dxdy &= \int\limits_c^d \left(\int\limits_a^b f(x,y)dx \right) dy = \int\limits_c^d dy \int\limits_a^b f(x,y)dx \end{aligned}$$

Izračunavanje - pravougaona oblast

Izvršićemo podelu T oblasti D na podoblasti

$$D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], 1 \le i \le m, 1 \le j \le n,$$

tako da je $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$ i

$$\lambda(T) = \max_{\substack{1 \le i \le m \\ 1 \le j \le n}} \sqrt{(\Delta x_i)^2 + (\Delta y_j)^2}.$$

Neka je $\lambda(T_x) = \max_{1 \le i \le m} \Delta x_i$ i $\lambda(T_y) = \max_{1 \le j \le n} \Delta y_j$. Izaberimo za svako $i \in \{1, \dots, m\}$ i $j \in \{1, \dots, n\}$ tačku $M_i(x_i, y_i)$. Prema definiciji dvostrukog integrala,

$$\begin{aligned}
\iint_D f(x,y)dxdy &= \lim_{\lambda(T)\to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_i)\Delta x_i \Delta y_j = \lim_{\substack{\lambda(T_x) \to 0 \\ \lambda(T_y) \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_i)\Delta x_i \Delta y_j \\
&= \lim_{\lambda(T_x)\to 0} \sum_{i=1}^{m} \left(\lim_{\lambda(T_y)\to 0} \sum_{j=1}^{n} f(x_i, y_j)\Delta y_j \right) \Delta x_i \\
&= \lim_{\lambda(T_x)\to 0} \sum_{i=1}^{m} \left(\int_c^d f(x_i, y)dy \right) \Delta x_i = \int_a^b \left(\int_c^d f(x,y)dy \right) dx
\end{aligned}$$

Izračunavanje - pravougaona oblast

Primer

Neka je $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 3, 0 \leq y \leq 1\}$. *Izračunati*

$$I = \iint\limits_D (xy + x^3)dxdy.$$

$$\begin{aligned} I &= \int\limits_1^3 dx \int\limits_0^1 (xy + x^3)dy = \int\limits_1^3 \left(\frac{1}{2}xy^2 + x^3y\right)\Big|_{y=0}^{y=1} dx \\ &= \int\limits_1^3 \frac{1}{2}x + x^3 dx = \left(\frac{1}{4}x^2 + \frac{1}{4}x^4\right)\Big|_1^3 = 22 \end{aligned}$$

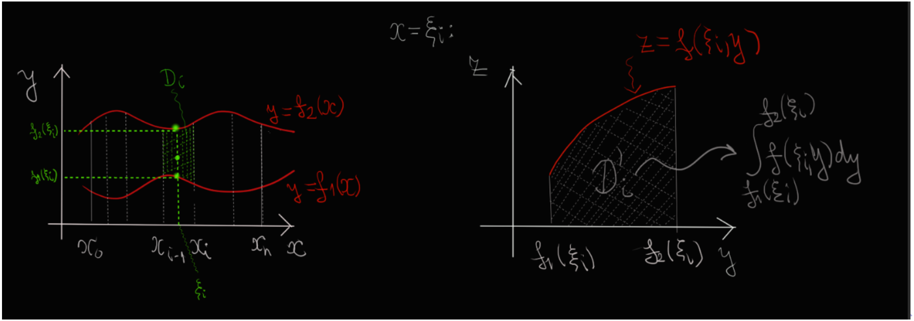
Izračunavanje - deo vertikalne trake

Neka su f_1 i f_2 neprekidne funkcije nad $[a, b]$, sa osobinom $f_1(x) \leq f_2(x)$ kada $x \in [a, b]$. Ako je funkcija $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^2$, neprekidna nad $D = \{(x, y) \in \mathbb{R}^2 : \ a \leq x \leq b, \ f_1(x) \leq y \leq f_2(x)\}$, onda je

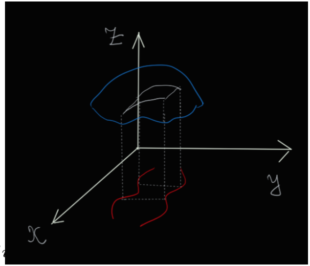
$$\iint\limits_D f(x,y)dxdy = \int\limits_a^b dx \int\limits_{f_1(x)}^{f_2(x)} f(x,y)dy = \int\limits_a^b \left(\int\limits_{f_1(x)}^{f_2(x)} f(x,y)dy \right) dx.$$

Izračunavanje - deo vertikalne trake

$$
\begin{aligned}
D &= \{D_1, \ldots, D_n\} : \\
& x = x_0 (= a), x = x_1, \ldots, x = x_{n-1}, x = x_n (= b) \qquad M_i(\xi_i, \zeta_i) \in D_i \\
D_i' &= \{(y, z) \in \mathbb{R}^2 : f_1(x_i) \le y \le f_2(x_i), 0 \le z \le f(x_i, y)\} \\
\Delta D_i' &= \int_{f_1(\xi_i)}^{f_2(\xi_i)} f(\xi_i, y) dy \quad \text{i} \quad s'(f, T) = \sum_{i=1}^{n} \Delta x_i \int_{f_1(\xi_i)}^{f_2(\xi_i)} f(\xi_i, y) dy.
\end{aligned}
$$



Izračunavanje - deo vertikalne trake



$$
\begin{aligned}
\iint\limits_D f(x,y)dxdy &= \lim_{\lambda(T)\to 0} \sum_{i=1}^{n} f(M_i)\Delta D_i \\
&= \lim_{\lambda(T)\to 0} \sum_{i=1}^{n} (\Delta D_i') \cdot \Delta x_i \\
&= \lim_{\lambda(T)\to 0} \sum_{i=1}^{n} \underbrace{\left(\int\limits_{f_1(\xi_i)}^{f_2(\xi_i)} f(\xi_i, y)dy \right)}_{\varphi(\xi_i)} \cdot \Delta x_i \\
&= \int\limits_a^b \left(\underbrace{\int\limits_{f_1(x)}^{f_2(x)} f(x,y)dy}_{\varphi(x)} \right) dx
\end{aligned}
$$

Izračunavanje - deo horizontalne trake

Neka su ψ_1 i ψ_2 neprekidne funkcije nad $[c,d]$, sa osobinom $\psi_1(y) \le \psi_2(y)$, $y \in [c,d]$. Ako je funkcija $f : \sigma \to \mathbb{R}$, $\sigma \subset \mathbb{R}^2$, neprekidna nad $D = \{(x,y) \in \mathbb{R}^2 : \; c \le y \le d, \; \psi_1(y) \le x \le \psi_2(y)\}$, gde su ψ_1 i ψ_2 neprekidne funkcije nad $[c,d]$, onda je

$$\boxed{\iint\limits_D f(x,y)dxdy = \int\limits_c^d dy \int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx.}$$

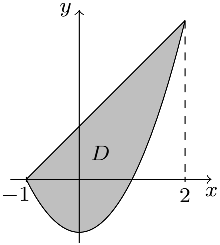
Primer

Primer

Izračunati $\iint\limits_D x\,dx\,dy$ *ako je*

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 - 1 \leq y \leq x + 1\}$$

$$\begin{aligned} & (y = x + 1 \ \wedge \ y = x^2 - 1) \\ \Leftrightarrow \quad & (y = x + 1 \ \wedge \ x^2 - x - 2 = 0) \\ \Leftrightarrow \quad & (y = x + 1 \ \wedge \ (x = -1 \vee x = 2)). \end{aligned}$$



Odatle zaključujemo da važi

$$D = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 2, x^2 - 1 \leq y \leq x + 1\}$$

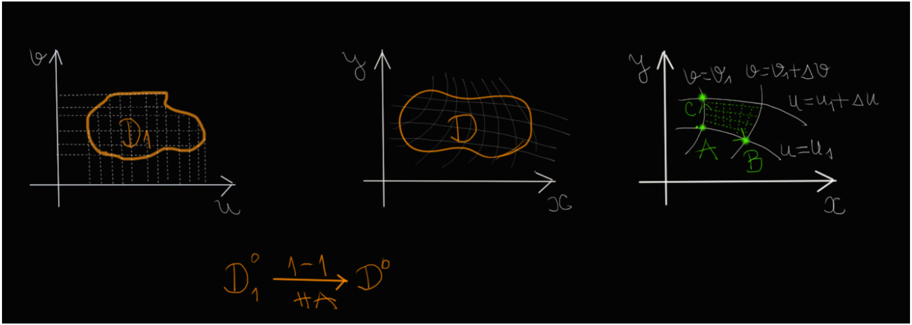
$$\iint\limits_D f(x, y)\,dx\,dy = \int\limits_{-1}^{2} dx \int\limits_{x^2-1}^{x+1} x\,dy = \int\limits_{-1}^{2} xy\Big|_{y=x^2-1}^{y=x+1} dx = \int\limits_{-1}^{2} (-x^2 + x^2 + 2x)\,dx = 3.$$

Transformacija koordinata

$$x = x(u,v), \quad y = y(u,v), \quad (u,v) \in D_1$$

$$A(x(u_1,v_1), y(u_1,v_1), 0), B(x(u_1,v_1+\Delta v), y(u_1,v_1+\Delta v), 0), C(x(u_1+\Delta u,v_1), y(u_1+\Delta u,v_1), 0)$$

$$\begin{aligned}
\overrightarrow{AB} &= (x(u_1,v_1+\Delta v) - x(u_1,v_1), y(u_1,v_1+\Delta v) - y(u_1,v_1), 0) \\
&\approx (x_v(u_1,v_1), y_v(u_1,v_1), 0)\Delta v \\
\overrightarrow{AC} &= (x(u_1+\Delta u,v_1) - x(u_1,v_1), y(u_1+\Delta u,v_1) - y(u_1,v_1), 0) \\
&\approx (x_u(u_1,v_1), y_u(u_1,v_1), 0)\Delta u
\end{aligned}$$



Transformacija koordinata

Površina paralelograma konstruisanog nad tim vektorima jednaka apsolutnoj vrednosti determinante

$$\left|\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x_v(u_1,v_1)\Delta v & y_v(u_1,v_1)\Delta v & 0 \\ x_u(u_1,v_1)\Delta u & y_u(u_1,v_1)\Delta u & 0 \end{vmatrix}\right| = \left|\begin{vmatrix} x_v(u_1,v_1) & y_v(u_1,v_1) \\ x_u(u_1,v_1) & y_u(u_1,v_1) \end{vmatrix}\Delta u \Delta v\right|$$

Prethodna determinanta se naziva Jakobijan transformacije i označava

$$J(u,v) \qquad \text{ili} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_v(u_1,v_1) & y_v(u_1,v_1) \\ x_u(u_1,v_1) & y_u(u_1,v_1) \end{vmatrix}$$

Vraćanjem posmatrane podele u integralu sumu, dobijamo

$$\boxed{\iint\limits_D f(x,y)dxdy = \iint\limits_{D_1} f(x(u,v),y(u,v))|J(u,v)|dudv.}$$

Primer

Primer

Izračunati dvostruki integral $I = \iint\limits_{\sigma} (x+y)dxdy$ *ako je*

$$\sigma = \{(x,y) \in \mathbb{R}^2 : \; y \leq x \leq 3y, \; 1 \leq x+y \leq 3\}.$$

$$\sigma = \{(x,y) \in \mathbb{R}^2 : \; 1 \leq \tfrac{x}{y} \leq 3, \; 1 \leq x+y \leq 3\}.$$

Smena: $u = \frac{x}{y}$, $v = x+y \Rightarrow x(u,v) = \frac{uv}{u+1}$, $y(u,v) = \frac{v}{u+1}$

$$J(u,v) = \begin{vmatrix} \frac{v}{(u+1)^2} & \frac{u}{u+1} \\ -\frac{v}{(u+1)^2} & \frac{1}{u+1} \end{vmatrix} = \frac{v}{(u+1)^2}$$

$$\begin{aligned} I &= \iint\limits_{\sigma} (x+y)dxdy \\ &= \iint\limits_{\sigma} (x(u,v)+y(u,v))|J(u,v)|dudv \\ &= \int\limits_1^3 \frac{1}{(u+1)^2} du \int\limits_1^3 v^2 dv = (-\tfrac{1}{u+1})\Big|_1^3 \cdot \tfrac{1}{3} v^3 \Big|_1^3 = \tfrac{13}{6}. \end{aligned}$$

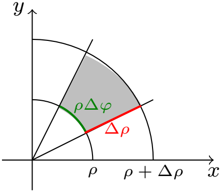
Transformacija koordinata - polarne koordinate

Veza Dekartovih i polarnih koordinata:

$$\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \end{aligned}$$

$$\rho \in [0,\infty), \qquad \varphi \in [0, 2\pi].$$

D_i :



$$\Delta D_i \approx \rho\Delta\varphi\Delta\rho$$

$$J(\rho,\varphi) = \begin{vmatrix} -\rho\sin\varphi & \rho\cos\varphi \\ \cos\varphi & \sin\varphi \end{vmatrix} = -\rho\sin^2\varphi - \rho\cos^2\varphi = -\rho$$

$$\boxed{\iint\limits_D f(x,y)dxdy = \iint\limits_{D_1} f(\rho\cos\varphi, \rho\sin\varphi)\rho d\rho d\varphi}.$$

Primer

Primer

Izračunati dvostruki integral $I = \iint\limits_{\sigma} (x^2 + y^2) dxdy$ *ako je*

$$\sigma = \{(x, y) \in \mathbb{R}^2 : \ 1 \leq x^2 + y^2 \leq 16, \ 0 < \frac{x}{\sqrt{3}} \leq y \leq \sqrt{3}x\}.$$

Prelaskom na polarne koordinate dobijamo

$$\begin{aligned} I &= \iint\limits_{\sigma} (x^2 + y^2) dxdy = \int\limits_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\varphi \int\limits_{1}^{4} \rho^3 d\rho \\ &= \varphi\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cdot \frac{1}{4}\rho^4\Big|_1^4 = \frac{32\pi}{3}. \end{aligned}$$

Primer

Primer

Izračunati površinu figure

$$D = \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{16} \leq 1,\ 0 \leq x \leq y\}.$$

Smena:

$$\begin{aligned} x &= 3\rho\cos\varphi \\ y &= 4\rho\sin\varphi \quad (\rho,\varphi) \in [0,1] \times [\tfrac{\pi}{4}, \pi] \end{aligned}$$

$$\begin{aligned} J(\rho,\varphi) &= \frac{\partial(x,y)}{\partial(\rho,\varphi)} = \begin{vmatrix} 3\cos\varphi & -3\rho\sin\varphi \\ 4\sin\varphi & 4\rho\cos\varphi \end{vmatrix} = 12\rho \\ \Delta D &= \int_{\frac{\pi}{4}}^{\pi} d\varphi \int_0^1 12\rho d\rho = \varphi|_{\frac{\pi}{4}}^{\pi} \cdot \tfrac{1}{2}\rho^2\Big|_0^1 = \frac{3\pi}{8}. \end{aligned}$$

Primena

Neka je $\mu(x, y)$ gustina ploče $D \subset \mathbb{R}^2$ u tački $(x, y) \in D$.
Masa m ploče $D \subset \mathbb{R}^2$ data je sa

$$m = \iint_D \mu(x, y) dxdy.$$

Težište $T(x_t, y_t)$ ploče $D \subset \mathbb{R}^2$ ima koordinate

$$x_t = \frac{1}{m} \iint_D x\mu(x, y) dxdy, \quad y_t = \frac{1}{m} \iint_D y\mu(x, y) dxdy.$$

Momenti inercije ploče $D \subset \mathbb{R}^2$ u odnosu na ose Ox i Oy su dati formulama

$$I_x = \iint_D y^2 \mu(x, y) dxdy, \quad I_y = \iint_D x^2 \mu(x, y) dxdy.$$

Moment inercije ploče $D \subset \mathbb{R}^2$ u odnosu na koordinatni početak dat je formulom

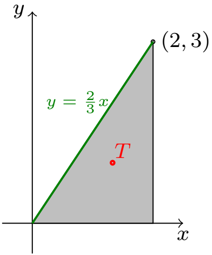
$$I_O = \iint_D (x^2 + y^2) \mu(x, y) dxdy.$$

Primer

Primer

Odrediti težište homogene ploče oblika pravouglog trougla čije su katete jednake 2 *i* 3.

$\rho(x,y) = \rho \quad (= \texttt{const})$



$$x_t = \frac{\iint\limits_D x\rho dD}{\iint\limits_D \rho dD} = \frac{\iint\limits_D x dD}{\iint\limits_D dD} = \frac{1}{\Delta D}\int\limits_0^2 x dx \int\limits_0^{\frac{3}{2}x} dy = \frac{4}{3},$$

$$y_t = \frac{\iint\limits_D y dD}{\iint\limits_D dD} = \frac{1}{3}\int\limits_0^3 y dy \int\limits_{\frac{2}{3}y}^2 dx = 1$$

Znači $T\left(\frac{4}{3}, 1\right)$.