MATEMATIKA 3

Ivan Prokić
kabinet 117, F blok
prokic@uns.ac.rs
http://imft.ftn.uns.ac.rs/∼iprokic/

Andrea Karalić
kabinet 215, F blok
andrea.karalic@gmail.com

Novi Sad

Slajdove pripremila prof. dr Jovanka Pantović

Tema 1

Trostruki integral

Definicija trostrukog integrala

Neka je funkcija $f : V \to \mathbb{R}$, $V \subset \mathbb{R}^3$, ograničena nad V, gde je V ograničena i zatvorena oblast.

Podela skupa V:

$$T = \{V_1, V_2, \dots, V_n\}$$

Izbor tačaka:

$$M_i \in V_i, \quad 1 \leq i \leq n$$

Parametar podele:

$$\mu(T) = \max_{1 \leq i \leq n} d(V_i).$$

Integralna suma:

$$s(f, T) = \sum_{i=1}^{n} f(M_i) \Delta V_i$$

$$\boxed{\iiint\limits_V f(x, y, z) dV = \iiint\limits_V f(x, y, z) dx dy dz = \lim_{\mu(T) \to 0} s(f, T)}$$

Trostruki integral - osobine

(1) $\iiint\limits_V \big(\alpha f(x,y,z) + \beta g(x,y,z)\big) dxdydz = \alpha \iiint\limits_V f(x,y,z) dxdydz + \beta \iiint\limits_V g(x,y,z) dxdydz,$ $\alpha, \beta \in \mathbb{R};$

(2) $\iiint\limits_V f(x,y,z) dxdydz = \iiint\limits_{V_1} f(x,y,z) dxdydz + \iiint\limits_{V_2} f(x,y,z) dxdydz$, gde je $V = V_1 \cup V_2$ i V_1 i V_2 nemaju zajedničkih unutrašnjih tačaka.

(3) $V = \{(x,y,z) \in \mathbb{R}^3 : \ (x,y) \in \sigma, \ z_1(x,y) \leq z \leq z_2(x,y)\}$. Tada je

$$\iiint\limits_V f(x,y,z) dxdydz = \iint\limits_\sigma \Big(\int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z) dz \Big) dxdy.$$

(4) $\Delta V = \iiint\limits_V dxdydz.$

Trostruki integral - transformacija koordinata

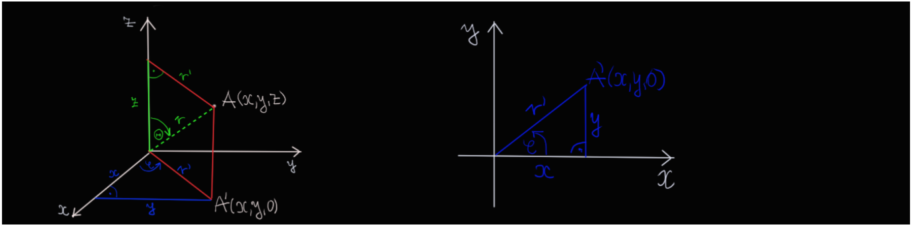
$$x = x(u,v,t), \quad y = y(u,v,t), \quad , z = z(u,v,t), \quad (u,v,t) \in V_1$$

Jakobijan transformacije:

$$J(u,v,t) = \frac{\partial(x,y,z)}{\partial(u,v,t)} = \begin{vmatrix} x_u & x_v & x_t \\ y_u & y_v & y_t \\ z_u & z_v & z_t \end{vmatrix} \neq 0, \quad (u,v,t) \in D_1^o,$$

$$\iiint\limits_{V} f(x,y,z)dxdydz = \iiint\limits_{V_1} f(x(u,v,t), y(u,v,t), z(u,v,t))|J(u,v,t)|dudvdt.$$

Sferne koordinate



$$\begin{aligned} \cos\theta &= \frac{z}{r} \\ \sin\theta &= \frac{r'}{r} \end{aligned}$$

$$\begin{aligned} x &= r'\cos\varphi = r\cos\varphi\sin\theta \\ y &= r'\sin\varphi = r\sin\varphi\sin\theta \\ z &= r\cos\theta \end{aligned}$$

$$r \geq 0, \quad \theta \in [0, \pi], \quad \varphi \in [0, 2\pi]$$

Sferne koordinate - Jakobijan

$$
\begin{aligned}
J &= \frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} \\
&= \begin{vmatrix} \cos\varphi\sin\theta & -r\sin\varphi\sin\theta & r\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} \\
&= r^2\sin\theta(-\cos^2\varphi\sin^2\theta - \sin^2\varphi\cos^2\theta - \cos^2\varphi\cos^2\theta - \sin^2\varphi\sin^2\theta) \\
&= -r^2\sin\theta\big(\sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \cos^2\theta(\sin^2\varphi + \cos^2\varphi)\big) \\
&= -r^2\sin\theta(\sin^2\theta + \cos^2\theta) = -r^2\sin\theta.
\end{aligned}
$$

Primena

Masa tela: $\mu = \mu(x, y, z)$ gustina tela

$$m = \iiint\limits_V \mu(x, y, z) dxdydz$$

Težište tela: $T(x_t, y_t, z_t)$

$$x_t = \frac{1}{m} \iiint\limits_V x\mu(x, y, z) dxdydz, y_t = \frac{1}{m} \iiint\limits_V y\mu(x, y, z) dxdydz, z_t = \frac{1}{m} \iiint\limits_V z\mu(x, y, z) dxdydz.$$

Moment inercije u odnosu na ravan yOz :

$$I_{yz} = \iiint\limits_V x^2 \mu(x, y, z) dxdydz.$$

Moment inercije u odnosu na osu Oz :

$$I_z = \iiint\limits_V (x^2 + y^2) \mu(x, y, z) dxdydz.$$

Moment inercije u odnosu na koordinatni početak:

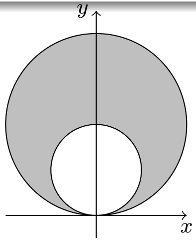
$$I_O = \iiint\limits_V (x^2 + y^2 + z^2) \mu(x, y, z) dxdydz.$$

Primer

Primer

Izračunati zapreminu tela

$$V = \{(x,y,z) \in \mathbb{R}^3 : \ 2y \leq x^2 + y^2 \leq 4y, \ -x^2 - y^2 \leq z \leq 0\}.$$



$$\begin{aligned} \sigma &= \{(x,y) \in \mathbb{R}^2 : 2y \leq x^2 + y^2 \leq 4y\} \\ &= \{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 \geq 1, x^2 + (y-2)^2 \leq 4\}. \end{aligned}$$

$$\begin{aligned} \Delta V &= 2 \iint_{\sigma} (x^2 + y^2) dx dy = 2 \int_0^{\frac{\pi}{2}} d\varphi \int_{2\sin\varphi}^{4\sin\varphi} \rho^3 d\rho \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \rho^4 \Big|_{2\sin\varphi}^{4\sin\varphi} d\varphi = 120 \int_0^{\frac{\pi}{2}} \sin^4 \varphi d\varphi \\ &= 120 \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2\varphi}{2}\right)^2 d\varphi \\ &= 30 \int_0^{\frac{\pi}{2}} \big(1 - 2\cos 2\varphi + \cos^2 2\varphi\big) d\varphi = \frac{7}{4}\pi. \end{aligned}$$

Primer

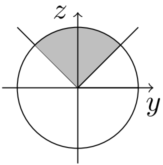
Primer

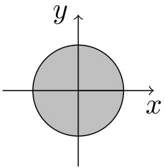
Izračunati zapreminu oblasti

$$V = \{(x,y,z) \in \mathbb{R}^3 : \ x^2+y^2+z^2 \leq 18, \ \sqrt{x^2+y^2} \leq z\}.$$

Presek oblasti:

$$\begin{array}{rcl} x^2+y^2+z^2 & = & 18 \\ \sqrt{x^2+y^2} & = & z \end{array} \Leftrightarrow \begin{array}{rcl} x^2+y^2 & = & 9 \\ \sqrt{x^2+y^2} & = & z. \end{array} \Leftrightarrow z = 3, x^2+y^2 = 9$$





Primer

Primer

Izračunati zapreminu oblasti

$$V = \{(x,y,z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 \leq 18, \ \sqrt{x^2+y^2} \leq z\}.$$

$$\sigma = \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq 9\}.$$

$$\begin{aligned}
\Delta V &= \iiint\limits_V dxdydz = \iint\limits_\sigma \left(\sqrt{18 - x^2 - y^2} - \sqrt{x^2+y^2}\right) dxdy \\
&= \int\limits_0^{2\pi} d\varphi \int\limits_0^3 \left(\sqrt{18-\rho^2} - \rho\right)\rho d\rho \\
&= \int\limits_0^{2\pi} d\varphi \int\limits_0^3 \sqrt{18-\rho^2}\rho d\rho - \int\limits_0^{2\pi} d\varphi \int\limits_0^3 \rho^2 d\rho = 54\pi(\sqrt{2}-1).
\end{aligned}$$