FTN, MAS, Fazi modeli odlučivanja, KOLOKVIJUM 1 **januar 2023.**

1. Neka je $X=\mathbb{R}$, $\mathcal{F}=\mathcal{P}(X)$, i neka je idempotentna (maksitivna) fazi-mera $\mu:\mathcal{F}\to[0,\infty]$ određena gustinom
$$\varphi=\begin{cases} 3 & , \quad x\le -2\\ -x & , \quad -2<x\le 0\\ \sqrt{2x-x^2} & , \quad 0<x\le 1\\ 1 & , \quad 1<x\end{cases}.$$
   (a) Izračunati $\mu(S)$ za skupove $S=A=\{-0.5,0,0.5\}$, $S=B=[-1,1]$ i $S=C=[-5,1]$.
   (b) Nad skupom $B=[-1,1]$ izračunati integral funkcije $f:\mathbb{R}\to[0,\infty)$ definisane sa $f(x)=|1-x|$, $x\in\mathbb{R}$.

2. Neka je funkcija $g:[0,\infty]\to[0,\infty]$ definisana sa $g(x)=x^2$, $x\in[0,\infty]$, i neka je
$$x\oplus y=g^{-1}(g(x)+g(y)),\qquad x\odot y=g^{-1}(g(x)\cdot g(y)),\qquad x,y\in[0,\infty].$$
   (a) Po $x,y\in[0,\infty]$ rešiti jednačinu $x^2\oplus(2\odot x)\oplus\mathbf{1}=\mathbf{0}$, gde su $\mathbf{0}$ i $\mathbf{1}$ redom neutralni elementi operacija $\oplus$ i $\odot$, i gde je $x^n\stackrel{\text{def}}{=}\underbrace{x\odot x\odot\ldots\odot x}_{n}$ za sve $n\in\mathbb{N}$ i $x\in[0,\infty]$.
   (b) Izračunati σ-$\oplus$-$\odot$ pseudo-integral funkcije
   $f:[0,\infty]\to[0,\infty],\ f(x)=\sqrt{x}e^x,\ x\in[0,\infty]$
   nad intervalom $[0,3]\in\mathcal{B}_{[0,\infty]}$, u odnosu na σ-$\oplus$-dekompozabilnu fazi meru $m:\mathcal{B}_{[0,\infty]}\to[0,\infty]$.

3. Neka je $I=[0,\infty]$, $\oplus=\sup$ i $\odot=\inf$. Neka je funkcija $\varphi:\mathbb{R}\to I$ definisana sa $\varphi(x)=\begin{cases}0 & , \quad x\le -2\\ x+2 & , \quad -2<x\le 0\\ -\frac{1}{2}x+2 & , \quad 0<x\le 4\\ 1 & , \quad 4<x\end{cases}$. Neka je skupovna funkcija $m:\mathcal{P}(\mathbb{R})\to I$ definisana sa $m(\emptyset)=0$ i $m(S)=\sup\limits_{x\in S}\varphi(x)$ za $S\in\mathcal{P}(\mathbb{R})$, $S\neq\emptyset$.
   (a) Za $S=[1,5]$ izračunati $m(S)$.
   (b) Za $S=[-3,5]$ i funkciju $f:\mathbb{R}\to I$ definisanu sa $f(x)=\begin{cases}0.5 & , \quad x\le 0\\ \frac{1}{4}x & , \quad 0<x\end{cases}$, izračunati $\int\limits_S^{\oplus} f\odot dm$.

FTN, MAS, Fazi modeli odlučivanja, KOLOKVIJUM 2 **januar 2023.**

1. Neka je $X=\{a,b,c\}$ i $\mathcal{F}=\mathcal{P}(X)$. Na merljivom prostoru $(X,\mathcal{P}(X))$, fazi-mera $\mu:\mathcal{P}(X)\to[0,\infty]$ u širem smislu je definisana sa $\mu(E)=\begin{cases}|E| & , \quad E\neq\{a,b\}\\ 3 & , \quad E=\{a,b\}\end{cases}$, $E\in\mathcal{P}(X)$. Neka je funkcija $f:X\to[0,\infty)$ definisana sa $f=\begin{pmatrix}a & b & c\\ 3 & 2.5 & 2\end{pmatrix}$. Izračunati Sugenov integral funkcije f nad skupom X.

2. Neka je $\omega_1=0.7$, $\omega_2=1$ i $\omega_3=0.4$, i neka je funkcija $A_{[3]}:[0,1]^3\to[0,1]$ definisana sa
$$A_{[3]}(a_1,a_2,a_3)=\min_{i\in\{1,2,3\}}\max(1-\omega_i,a_i),\qquad a_1,a_2,a_3\in[0,1].$$
Dokazati da je $A_{[3]}$ funkcija agregacije, i ispitati da li je simetrična.

3. Neka je $p\ge 1$, $q\ge 1$, i neka za $\omega_1,\omega_2\in[0,1]$ važi $\omega_1+\omega_2=1$. Neka su metrike $d_p:\mathbb{R}^2\times\mathbb{R}^2\to[0,\infty)$ i $d_q:\mathbb{R}^2\times\mathbb{R}^2\to[0,\infty)$ definisane sa
$$d_p((x_1,y_1),(x_2,y_2))=(|x_1-x_2|^p+|y_1-y_2|^p)^{\frac{1}{p}},\qquad (x_1,y_1),(x_2,y_2)\in\mathbb{R}^2,$$
$$d_q((x_1,y_1),(x_2,y_2))=(|x_1-x_2|^q+|y_1-y_2|^q)^{\frac{1}{q}},\qquad (x_1,y_1),(x_2,y_2)\in\mathbb{R}^2,$$
i neka je funkcija agregacije $\mathsf{WAM}_{[2]}:[0,\infty)\to[0,\infty)$ definisana sa
$$\mathsf{WAM}_{[2]}(a,b)=\omega_1 a+\omega_2 b,\qquad a,b\in[0,\infty).$$
Neka je funkcija $d:\mathbb{R}^2\times\mathbb{R}^2\to[0,\infty)$ definisana sa
$$d((x_1,y_1),(x_2,y_2))=\mathsf{WAM}_{[2]}(d_p((x_1,y_1),(x_2,y_2)),d_q((x_1,y_1),(x_2,y_2)))$$
za sve $(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$.
   (a) Dokazati da je funkcija d metrika.
   (b) Ispitati da li je funkcija agregacije $\mathsf{WAM}_{[2]}$ idempotentna.

1. (a) $\mu(A)=\sup\limits_{x\in X}\varphi(x)=\max\{\varphi(-0.5),\varphi(0),\varphi(0.5)\}=\max\left\{0.5,0,\sqrt{2\cdot 0.5-0.5^2}\right\}=\max\left\{0.5,0,\sqrt{0.75}\right\}=\sqrt{0.75}$

jer je $\sqrt{0.75}>\sqrt{0.5}>0.5$,

$$\mu(B)=\sup_{x\in B}\varphi(x)=\sup_{x\in[-1,1]}\varphi(x)=\max\left\{\sup_{x\in[-1,0]}\varphi(x),\ \sup_{x\in(0,1]}\varphi(x)\right\}=\max\left\{\sup_{x\in[-1,0]}-x,\ \sup_{x\in(0,1]}\sqrt{2x-x^2}\right\}$$

$$=\max\left\{1,\ \sup_{x\in(0,1]}\sqrt{2\cdot 1-1^2}\right\}=\max\{1,1\}=1$$

jer je $-x$, $x\in[-1,0]$ opadajuća funkcija koja supremum dostiže u levom kraju intervala $[-1,0]$, a $\sqrt{2x-x^2}$, $x\in(0,1]$ je rastuća funkcija koja supremum dostiže u desnom kraju intervala $(0,1]$. Naime, funkcija $\varphi_1(x)=\sqrt{2x-x^2}$, $x\in(0,1]$ je rastuća jer je $\varphi_1'(x)=\dfrac{2-2x}{2\sqrt{2x-x^2}}=\dfrac{1-x}{\sqrt{2x-x^2}}\geq 0$, $x\in(0,1]$.

(b) Kako je

$$f(x)=|1-x|=\begin{cases}1-x & , \ 1-x\geq 0\\ -(1-x) & , \ 1-x<0\end{cases}=\begin{cases}1-x & , \ x\leq 1\\ x-1 & , \ x>1\end{cases},$$

sledi da je $f(x)=1-x$, $x\in[-1,1]$. Dobijamo

$$\bigvee_B f d\mu=\bigvee_{[-1,1]} f d\mu=\sup_{x\in[-1,1]}\{f(x)\cdot\varphi(x)\}=\sup\{f(x)\cdot\varphi(x)\mid x\in[-1,1]\}$$

$$=\max\left\{\sup_{x\in[-1,0]}\{f(x)\cdot\varphi(x)\},\ \sup_{x\in(0,1]}\{f(x)\cdot\varphi(x)\}\right\}$$

$$=\max\left\{\sup_{x\in[-1,0]}\{(1-x)\cdot(-x)\},\ \sup_{x\in(0,1]}\left\{(1-x)\cdot\sqrt{2x-x^2}\right\}\right\}$$

$$=\max\left\{\sup_{x\in[-1,0]}\left\{x^2-x\right\},\ \sup_{x\in(0,1]}\left\{(1-x)\cdot\sqrt{2x-x^2}\right\}\right\}.$$

Za funkciju $g_1(x)=x^2-x$, $x\in[-1,0]$ je $g_1'(x)=2x-1<0$, $x\in[-1,0]$. Sledi da je ona opadajuća na intervalu $[-1,0]$, te supremum dostiže u levom kraju intervala. Dakle, $\sup\limits_{x\in[-1,0]}\left\{x^2-x\right\}=(-1)^2-(-1)=2$.

Za funkciju $g_2(x)=(1-x)\cdot\sqrt{2x-x^2}$, $x\in(0,1]$ dobijamo da je njen izvod

$$g_2'(x)=-\sqrt{2x-x^2}+(1-x)\cdot\frac{2-2x}{2\sqrt{2x-x^2}}=-\sqrt{2x-x^2}+\frac{(1-x)^2}{\sqrt{2x-x^2}}=\frac{-\left(2x-x^2\right)+(1-x)^2}{\sqrt{2x-x^2}}=\frac{2x^2-4x+1}{\sqrt{2x-x^2}}$$

za $x\in(0,1]$. Dalje je

$$g_2'(x)=0\quad\Leftrightarrow\quad 2x^2-4x+1=0\quad\Leftrightarrow\quad x_{1,2}=\frac{4\pm\sqrt{16-8}}{4}=\frac{4\pm 2\sqrt{2}}{4}=1\pm\frac{\sqrt{2}}{2},$$

gde je $1+\dfrac{\sqrt{2}}{2}>1$, a $1-\dfrac{\sqrt{2}}{2}\in(0,1]$ je stacionarna tačka na intervalu $(0,1]$. Stoga je

$$\sup_{x\in(0,1]}g_2(x)=\sup_{x\in(0,1]}\left\{(1-x)\cdot\sqrt{2x-x^2}\right\}=\max\left\{g_2(0),g_2\left(1-\frac{\sqrt{2}}{2}\right),g_2(1)\right\}$$

$$=\max\left\{0,g_2\left(1-\frac{\sqrt{2}}{2}\right),0\right\}=g_2\left(1-\frac{\sqrt{2}}{2}\right)$$

jer je $f(x)\geq 0$, $\varphi(x)\geq 0$, te i $f(x)\cdot\varphi(x)\geq 0$ za sve $x\in X=\mathbb{R}$. Pri tome je

$$g_2\left(1-\frac{\sqrt{2}}{2}\right)=\left(1-\left(1-\frac{\sqrt{2}}{2}\right)\right)\cdot\sqrt{2\left(1-\frac{\sqrt{2}}{2}\right)-\left(1-\frac{\sqrt{2}}{2}\right)^2}$$

$$=\frac{\sqrt{2}}{2}\cdot\sqrt{2-\sqrt{2}-\left(1-\sqrt{2}+\frac{1}{2}\right)}=\frac{\sqrt{2}}{2}\cdot\sqrt{1-\frac{1}{2}}=\frac{1}{2}.$$

Sledi da je ona opadajuća na intervalu $(0,1]$, te supremum dostiže u levom kraju intervala. Dakle,

$$\sup_{x\in(0,1]}g_2(x)=g_2\left(1-\frac{\sqrt{2}}{2}\right)=\frac{1}{2},$$

te je

$$\bigvee_B f d\mu=\max\left\{\sup_{x\in[-1,0]}\left\{x^2-x\right\},\ \sup_{x\in(0,1]}\left\{(1-x)\cdot\sqrt{2x-x^2}\right\}\right\}=\max\left\{2,\frac{1}{2}\right\}=2.$$

2. Inverzna funkcija funkcije g je gunkcija $g^{-1} : [0,\infty] \to [0,\infty]$ definisana sa $g^{-1}(x) = \sqrt{x}$, $x \in [0,\infty]$. Sledi da je

$$x \oplus y = g^{-1}(g(x) + g(y)) = \sqrt{x^2 + y^2}, \qquad x \odot y = g^{-1}(g(x) \cdot g(y)) = \sqrt{x^2 \cdot y^2} = xy \qquad x, y \in [0,\infty].$$

Dakle, za $I = [0,\infty]$ je $(I, \oplus, \odot)$ poluprsten tipa [PPT2b] čije su operacije $\oplus$ i $\odot$ generisane rastućom funkcijom g. U ovom slučaju je $\mathbf{0} = 0$ jer za sve $x \in I$ važi $\mathbf{0} \oplus x = 0 \oplus x = \sqrt{0^2 + x^2} = x$ i takođe $x \oplus \mathbf{0} = x$. Relacija $\preceq$ je pri tome standardna relacija $\leq$ te je $I_+ = \{x \in I \mid \mathbf{0} \preceq x\} = \{x \in I \mid 0 \leq x\} = I$. Kako je $x \odot y = xy$, $x, y \in I = [0,\infty]$, sledi da je $\mathbf{1} = 1$.

(a) Kako za sve $x \in [0,\infty]$ važi $a = x^2 \in \mathbb{R}$, $b = 2 \odot x \in \mathbb{R}$, $\mathbf{0} = 0$ i $\mathbf{1} = 1$, dobijamo da je

$$x^2 \oplus (2 \odot x) \oplus \mathbf{1} = \mathbf{0} \quad \Leftrightarrow \quad a \oplus b \oplus 1 = 0 \quad \Leftrightarrow \quad \sqrt{a^2 + b^2} \oplus 1 = 0$$

$$\Leftrightarrow \quad \sqrt{\left(\sqrt{a^2 + b^2}\right)^2 + 1^2} = 0 \quad \Leftrightarrow \quad \sqrt{a^2 + b^2 + 1} = 0 \quad \Leftrightarrow \quad a^2 + b^2 + 1 = 0,$$

te je jasno da polazna jednačina nema rešenja u skupu $I = [0,\infty]$.

(b) Funkcija $f : [0,\infty] \to [0,\infty]$, $f(x) = \sqrt{x}e^x$, $x \in [0,\infty]$ je očigledno merljiva odnosno integrabilna nad $[0,3]$. Pri tome je $\lambda = g \circ m : \mathcal{B}_{[0,\infty]} \to [0,\infty]$ Lebegova mera na $\mathcal{B}_{[0,\infty]}$, te dobijamo

$$\int_{[0,3]}^{\oplus} f \odot dm = g^{-1}\left(\int_{[0,3]} (g \circ f) \cdot d\lambda\right) = g^{-1}\left(\int_0^3 g(f(\lambda)) d\lambda\right) = g^{-1}\left(\int_0^3 g\left(\sqrt{\lambda}e^{\lambda}\right) d\lambda\right) = g^{-1}\left(\int_0^3 \lambda e^{2\lambda} d\lambda\right).$$

Primenom formule za parcijalnu integraciju sa $u = \lambda$, $du = d\lambda$ i $dv = e^{2\lambda}d\lambda$, gde uz smenu $2\lambda = t$, $d\lambda = \frac{1}{2}dt$ dobijamo $v = \int e^{2\lambda} d\lambda = \frac{1}{2}\int e^t dt = \frac{1}{2}e^t = \frac{1}{2}e^{2\lambda}$, sledi

$$\int_0^3 \lambda e^{2\lambda} d\lambda = \frac{1}{2}\lambda e^{2\lambda}\Big|_0^3 - \frac{1}{2}\int_0^3 e^{2\lambda} d\lambda = \left(\frac{3}{2}e^6 - 0\right) - \frac{1}{4}e^{2\lambda}\Big|_0^3 = \frac{3}{2}e^6 - \left(\frac{1}{4}e^6 - \frac{1}{4}\right) = \frac{1}{4} + \frac{5}{4}e^6.$$

Time konačno dobijamo

$$\int_{[0,3]}^{\oplus} f \odot dm = g^{-1}\left(\int_0^3 \lambda e^{2\lambda} d\lambda\right) = g^{-1}\left(\frac{1}{4} + \frac{5}{4}e^6\right) = \sqrt{\frac{1}{4} + \frac{5}{4}e^6}.$$

3. Uređena trojka $(I, \sup, \inf)$ je poluprsten tipa [PPT3] gde je $\mathbf{0} = 0$, $\mathbf{1} = \infty$ i $I_+ = I = [0,\infty]$, a funkcija m je maksitivna fazi-mera sa vrednostima u poluprstenu $(I, \sup, \inf)$.

(a) $m([1,5]) = \sup\limits_{x \in [1,5]} \varphi(x) = \max\left\{\sup\limits_{x \in [1,4]} \varphi(x), \sup\limits_{x \in [4,5]} \varphi(x)\right\} = \max\left\{\sup\left\{-\frac{1}{2}x + 2 \;\middle|\; x \in [1,4]\right\}, \sup\limits_{x \in [4,5]} 1\right\}.$

Očigledno je $\sup\limits_{x \in [4,5]} 1 = 1$, a $h(x) = -\frac{1}{2}x + 2$, $x \in [1,4]$ je monotono opadajuća funkcija na $[1,4]$ koja svoj maksimum (supremum) dostiže u levom kraju intervala, te je $\sup\left\{-\frac{1}{2}x + 2 \;\middle|\; x \in [1,4]\right\} = h(1) = \frac{3}{2}$. Sledi da je

$$m([1,5]) = \max\left\{\frac{3}{2}, 1\right\} = \frac{3}{2}.$$

(b) $\displaystyle\int_{[-3,5]}^{\oplus} f \odot dm = \sup\{f(x) \wedge \varphi(x) \mid x \in [-3,5]\} = \max\{b_1, b_2, b_3, b_4\}$,

gde je

$b_1 = \sup\{f(x) \wedge \varphi(x) \mid x \in [-3,-2]\}$, $\qquad b_2 = \sup\{f(x) \wedge \varphi(x) \mid x \in (-2,0]\}$,

$b_3 = \sup\{f(x) \wedge \varphi(x) \mid x \in (0,4]\}$, $\qquad b_4 = \sup\{f(x) \wedge \varphi(x) \mid x \in (4,5]\}$.

(b.1) $b_1 = \sup\{f(x) \wedge \varphi(x) \mid x \in [-3,-2]\} = \sup\limits_{x \in [-3,-2]} \min\{f(x), \varphi(x)\} = \sup\limits_{x \in [-3,-2]} \min\{0.5, 0\} = \sup\limits_{x \in [-3,-2]} 0 = 0.$

(b.2) $b_2 = \sup\{f(x) \wedge \varphi(x) \mid x \in (-2,0]\} = \sup\limits_{x \in (-2,0]} \min\{f(x), \varphi(x)\} = \sup\limits_{x \in (-2,0]} \min\{0.5, x+2\}$

$$= \max\left\{\sup_{x \in (-2,-1.5]} \min\{0.5, x+2\}, \sup_{x \in (-1.5,0]} \min\{0.5, x+2\}\right\}$$

$$= \max\left\{\sup_{x \in (-2,-1.5]} \{x+2\}, \sup_{x \in (-1.5,0]} 0.5\right\} = \max\{0.5, 0.5\} = 0.5.$$

(b.3) $b_3 = \sup\{ f(x) \wedge \varphi(x) \mid x \in (0,4]\} = \sup\limits_{x\in(0,4]} \min\{f(x),\varphi(x)\} = \sup\limits_{x\in(0,4]} \min\left\{\frac{1}{4}x, -\frac{1}{2}x+2\right\}.$

Kako na $(0,4]$ važi

$\frac{1}{4}x \le -\frac{1}{2}x+2 \quad\Leftrightarrow\quad x \le -2x+8 \quad\Leftrightarrow\quad 3x \le 8 \quad\Leftrightarrow\quad x \le \frac{8}{3} \in (0,4],$

dalje dobijamo

$$b_3 = \max\left\{ \sup_{x\in\left(0,\frac{8}{3}\right]} \min\left\{\frac{1}{4}x, -\frac{1}{2}x+2\right\}, \sup_{x\in\left(\frac{8}{3},4\right]} \min\left\{\frac{1}{4}x, -\frac{1}{2}x+2\right\}\right\} = \max\left\{ \sup_{x\in\left(0,\frac{8}{3}\right]} \left\{\frac{1}{4}x\right\}, \sup_{x\in\left(\frac{8}{3},4\right]} \left\{-\frac{1}{2}x+2\right\}\right\}.$$

Funkcija $h_1(x) = \frac{1}{4}x$, $x \in \left(0,\frac{8}{3}\right]$ je monotono rastuća te svoj supremum dostiže u desnom kraju intervala, dakle

$$\sup_{x\in\left(0,\frac{8}{3}\right]} \left\{\frac{1}{4}x\right\} = h_1\left(\frac{8}{3}\right) = 6.$$

Funkcija $h_2(x) = -\frac{1}{2}x+2$, $x \in \left(\frac{8}{3},4\right]$ je monotono opadajuća te svoj supremum dostiže u levom kraju intervala, dakle

$$\sup_{x\in\left(\frac{8}{3},4\right]} \left\{-\frac{1}{2}x+2\right\} = h_2\left(\frac{8}{3}\right) = \frac{2}{3}.$$

Sledi da je

$$b_3 = \max\left\{6, \frac{2}{3}\right\} = 6.$$

(b.4) $b_4 = \sup\{ f(x) \wedge \varphi(x) \mid x \in (4,5]\} = \sup\limits_{x\in(4,5]} \min\{f(x),\varphi(x)\} = \sup\limits_{x\in(4,5]} \min\left\{\frac{1}{4}x, 1\right\}.$

Kako je $\frac{1}{4}x \ge 1$ za sve $x \in (4,5]$, dobijamo

$$b_4 = \sup_{x\in(4,5]} 1 = 1.$$

Konačno dobijamo

$$\int\limits_{[-3,5]}^{\oplus} f \odot dm = \max\{b_1,b_2,b_3,b_4\} = \max\{0, 0.5, 6, 1\} = 6.$$

REŠENJA - KOLOKVIJUM 2

1. Za $\alpha \in [0,\infty]$ je: $\quad F_\alpha(f) = \{x \in X \mid f(x) \ge \alpha\} = \begin{cases} X & , \ \alpha \in [0,2] \\ \{a,b\} & , \ \alpha \in (2,2.5] \\ \{a\} & , \ \alpha \in (2.5,3] \\ \emptyset & , \ \alpha \in (3,\infty] \end{cases}.$

Primenom definicije Sugenovog integrala dobijamo

$$(\mathrm{S})\int_X f d\mu = \{\alpha \wedge \mu(F_\alpha(f)) \mid \alpha \in [0,\infty]\} = \max\left\{X_{[0,2]}, X_{(2,2.5]}, X_{(2.5,3]}, X_{(3,\infty]}\right\},$$

gde je

$X_{[0,2]} = \sup\{\alpha \wedge \mu(F_\alpha(f)) \mid \alpha \in [0,2]\} = \sup\{\alpha \wedge \mu(X) \mid \alpha \in [0,2]\} = \sup\{\alpha \wedge 3 \mid \alpha \in [0,2]\} = \sup\{\alpha \mid \alpha \in [0,2]\} = 2,$

$X_{(2,2.5]} = \sup\{\alpha \wedge \mu(F_\alpha(f)) \mid \alpha \in (2,2.5]\} = \sup\{\alpha \wedge \mu(\{a,b\}) \mid \alpha \in (2,2.5]\} = \sup\{\alpha \wedge 3 \mid \alpha \in (2,2.5]\}$
$= \sup\{\alpha \mid \alpha \in (2,2.5]\} = 2.5,$

$X_{(2.5,3]} = \sup\{\alpha \wedge \mu(F_\alpha(f)) \mid \alpha \in (2.5,3]\} = \sup\{\alpha \wedge \mu(\{a\}) \mid \alpha \in (2.5,3]\} = \sup\{\alpha \wedge 1 \mid \alpha \in (2.5,3]\}$
$= \sup\{1 \mid \alpha \in (2.5,3]\} = 1,$

$X_{(3,\infty]} = \sup\{\alpha \wedge \mu(F_\alpha(f)) \mid \alpha \in (3,\infty]\} = \sup\{\alpha \wedge \mu(\emptyset) \mid \alpha \in (3,\infty]\} = \sup\{\alpha \wedge 0 \mid \alpha \in (3,\infty]\}$
$= \sup\{0 \mid \alpha \in (3,\infty]\} = 0,$

dakle: $\quad (\mathrm{S})\int_X f d\mu = \max\{2, 2.5, 1, 0\} = 2.5.$

2. U razvijenom obliku je

$A_{[3]}(a_1,a_2,a_3) = \min(\max(1-\omega_1,a_1),\max(1-\omega_2,a_2),\max(1-\omega_3,a_3)) = \min(\max(0.3,a_1),\max(0,a_2),\max(0.6,a_3))$

za sve $a_1,a_2,a_3 \in [0,1]$.

(a01) Granični uslovi važe jer je

$A_{[3]}(0,0,0) = \min(\max(0.3,0),\max(0,0),\max(0.6,0)) = \min(0.3,0,0.6) = 0,$

$A_{[3]}(1,1,1) = \min(\max(0.3,1),\max(0,1),\max(0.6,1)) = \min(1,1,1) = 1.$

(a02) Funkcija $A_{[3]}$ je monotono neopadajuća po svim komponentama jer za sve $a_1,a_2,a_3,b_1,b_2,b_3 \in [0,1]$ važi

$$(a_1 \le b_1 \ \wedge\ a_2 \le b_2 \ \wedge\ a_3 \le b_3)$$
$$\Rightarrow \begin{cases} A_1 = \max(0.3,a_1) \le \max(0.3,b_1) = B_1, \\ A_2 = \max(0,a_2) \le \max(0,b_2) = B_2, \\ A_3 = \max(0.6,a_3) \le \max(0.6,b_3) = B_3, \end{cases}$$
$$\Rightarrow \min(A_1,A_2,A_3) \le \min(B_1,B_2,B_3)$$
$$\Rightarrow A_{[3]}(a_1,a_2,a_3) \le A_{[3]}(b_1,b_2,b_3).$$

(a04) Funkcija agregacije $A_{[3]}$ nije simetrična funkcija jer na primer za vrednosti argumenata $(a_1,a_2,a_3) = (0.1,0.2,0.3) \in [0,1]^3$ i njihovu permutaciju $(a_1',a_2',a_3') = (0.2,0.3,0.1)$ dobijamo

$A_{[3]}(0.1,0.2,0.3) = \min(\max(0.3,0.1),\max(0,0.2),\max(0.6,0.3)) = \min(0.3,0.2,0.6) = 0.2,$

$A_{[3]}(0.2,0.3,0.1) = \min(\max(0.3,0.2),\max(0,0.3),\max(0.6,0.1)) = \min(0.3,0.3,0.6) = 0.3,$

dakle $A_{[3]}(0.1,0.2,0.3) \ne A_{[3]}(0.2,0.3,0.1)$.

3. (a) Kako su funkcije d_p i d_q metrike, za njih važi

(1) $\forall (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2,$
$d_p((x_1,y_1),(x_2,y_2)) = 0 \ \Leftrightarrow\ (x_1,y_1) = (x_2,y_2),$
$d_q((x_1,y_1),(x_2,y_2)) = 0 \ \Leftrightarrow\ (x_1,y_1) = (x_2,y_2),$

(2) $\forall (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2,$
$d_p((x_1,y_1),(x_2,y_2)) = d_p((x_2,y_2),(x_1,y_1)),$
$d_q((x_1,y_1),(x_2,y_2)) = d_q((x_2,y_2),(x_1,y_1)),$

(3) $\forall (x_1,y_1),(x_2,y_2),(x_3,y_3) \in \mathbb{R}^2,$
$d_p((x_1,y_1),(x_3,y_3)) \le d_p((x_1,y_1),(x_2,y_2)) + d_p((x_2,y_2),(x_3,y_3)),$
$d_q((x_1,y_1),(x_3,y_3)) \le d_q((x_1,y_1),(x_2,y_2)) + d_q((x_2,y_2),(x_3,y_3)),$

a za funkciju agregacije $\mathsf{WAM}_{[2]}$ važi

(4) $\mathsf{WAM}_{[2]}(0,0) = 0,$

(5) zbog $\omega_1,\omega_2 \in [0,1]$, $\omega_1+\omega_2 = 1$ je $\omega_1 > 0$ i/ili $\omega_2 > 0$,

(6) za $a_1 \le a_2$ i $b_1 \le b_2$ je $\mathsf{WAM}_{[2]}(a_1,b_1) \le \mathsf{WAM}_{[2]}(a_2,b_2)$.

Dokažimo da za funkciju d važe aksiome metrike.

(a.1) Za svako $(x_1,y_1) \in \mathbb{R}^2$ je

$$d((x_1,y_1),(x_1,y_1)) = \mathsf{WAM}_{[2]}(d_p((x_1,y_1),(x_1,y_1)),d_q((x_1,y_1),(x_1,y_1))) \overset{(1)}{=} \mathsf{WAM}_{[2]}(0,0) \overset{(4)}{=} 0.$$

(a.2) Za sve $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$ važi

$$d((x_1,y_1),(x_2,y_2)) = 0$$
$$\Rightarrow \mathsf{WAM}_{[2]}(d_p((x_1,y_1),(x_2,y_2)),d_q((x_1,y_1),(x_2,y_2))) = 0$$
$$\Rightarrow \omega_1 d_p((x_1,y_1),(x_2,y_2)) + \omega_2 d_q((x_1,y_1),(x_2,y_2)) = 0$$
$$\overset{(5)}{\Rightarrow} d_p((x_1,y_1),(x_2,y_2)) = 0 \ \vee\ d_q((x_1,y_1),(x_2,y_2)) = 0$$
$$\overset{(1)}{\Rightarrow} (x_1,y_1) = (x_2,y_2).$$

(a.3) Za sve $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$ važi

$$d((x_1,y_1),(x_2,y_2)) = \mathsf{WAM}_{[2]}(d_p((x_1,y_1),(x_2,y_2)),d_q((x_1,y_1),(x_2,y_2)))$$
$$\overset{(2)}{=} \mathsf{WAM}_{[2]}(d_p((x_2,y_2),(x_1,y_1)),d_q((x_2,y_2),(x_1,y_1))) = d((x_2,y_2),(x_1,y_1)).$$

(a.4) Neka je $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in \mathbb{R}^2$. Za

$a_1 = d_p((x_1,y_1),(x_3,y_3)), \qquad a_2 = d_p((x_1,y_1),(x_2,y_2)) + d_p((x_2,y_2),(x_3,y_3)),$

$b_1 = d_q((x_1,y_1),(x_3,y_3)), \qquad b_2 = d_q((x_1,y_1),(x_2,y_2)) + d_q((x_2,y_2),(x_3,y_3)),$

zbog (3) važi $a_1 \le a_2$ i $b_1 \le b_2$, te je

$$d((x_1,y_1),(x_3,y_3)) = \mathsf{WAM}_{[2]}(d_p((x_1,y_1),(x_3,y_3)),d_q((x_1,y_1),(x_3,y_3)))$$
$$\overset{(6)}{\le} \mathsf{WAM}_{[2]}(d_p((x_1,y_1),(x_2,y_2)) + d_p((x_2,y_2),(x_3,y_3)),\ d_q((x_1,y_1),(x_2,y_2)) + d_q((x_2,y_2),(x_3,y_3)))$$
$$= \omega_1(d_p((x_1,y_1),(x_2,y_2)) + d_p((x_2,y_2),(x_3,y_3))) + \omega_2(d_q((x_1,y_1),(x_2,y_2)) + d_q((x_2,y_2),(x_3,y_3)))$$
$$= \omega_1 d_p((x_1,y_1),(x_2,y_2)) + \omega_2 d_q((x_1,y_1),(x_2,y_2)) + \omega_1 d_p((x_2,y_2),(x_3,y_3)) + \omega_2 d_q((x_2,y_2),(x_3,y_3))$$
$$= \mathsf{WAM}_{[2]}(d_p((x_1,y_1),(x_2,y_2)),d_q((x_1,y_1),(x_2,y_2))) + \mathsf{WAM}_{[2]}(d_p((x_2,y_2),(x_3,y_3)),d_q((x_2,y_2),(x_3,y_3)))$$
$$= d((x_1,y_1),(x_2,y_2)) + d((x_2,y_2),(x_3,y_3)),$$

čime je za funkciju d dokazana nejednakost trougla.

(b) Funkcija agregacije $\mathsf{WAM}_{[2]}$ jeste idempotentna jer za svako $a, \in [0,\infty)$ zbog $\omega_1+\omega_2 = 1$ sledi

$\mathsf{WAM}_{[2]}(a,a) = \omega_1 a + \omega_2 a = (\omega_1+\omega_2)a = a.$