1. Primenom rednog algoritma sa pragom $\theta = 1$ odrediti jednu bazu skupa binarnih obeležja $S = \{p_1, p_2, p_3\}$, $p_1 = (100)$, $p_2 = (101)$, $p_3 = (001)$.

2. Na vektorskom prostoru $(\mathbb{R}^n, \mathbb{R}, +, \cdot)$ je funkcija $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ definisana sa

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_{i \in \{1,\ldots,n\}} |x_i - y_i|.$$

Dokazati da je funkcija d metrika na skupu $\mathbb{R}^n$, tj. da važi

(d01) $\forall x \in \mathbb{R}^n,\ d(x,y) = 0 \Leftrightarrow x = y$,

(d02) $\forall x, y \in \mathbb{R}^n,\ d(x,y) = d(y,x)$,

(d03) $\forall x, y, z \in \mathbb{R}^n,\ d(x,z) \le d(x,y) + d(y,z)$.

3. U $\mathbb{R}^3$, primenom rešavajućih funkcija

$d_1(x,y,z) = 2x - 4y - z + 5, \qquad d_2(x,y,z) = 3x + y - 6z + 2, \qquad d_3(x,y,z) = -2x - 3y + z - 1,$

po principu maksimuma (treći način), klasifikovati tačke

$A_1(-2,1,1)$, $A_2(8,3,3)$, $A_3(2,-5,-1)$, $A_4(2,2,2)$,
$A_5(-4,6,-2)$, $A_6(1,-7,-5)$, $A_7(-1,-2,-6)$, $A_8(8,-5,-2)$.

4. U vektorskom prostoru $\mathbb{R}^2$, klase ω_1, ω_2 i ω_3 su zadane svojim poredstavnicima

$z_1 = (-6,2) \in \omega_1, \quad z_2 = (3,2) \in \omega_2, \quad z_3 = (2,-5) \in \omega_3.$

Klasifikovanjem na osnovu minimuma rastojanja (primenom jednostrukog etalona) klasifikovati tačke

$A_1(-2,1)$, $A_2(8,3)$, $A_3(2,-5)$, $A_4(2,2)$,
$A_5(-4,6)$, $A_6(1,-7)$, $A_7(-1,-2)$, $A_8(8,-5)$.

Napisati jednačine razdelnih pravih klasa ω_1, ω_2 i ω_3.

REŠENJA

1. Sa $N = |S| = 3$ i početnim vrednostima $f_1(0) = (111)$, $f_2(0) = (111)$, $f_3(0) = (111)$ dobijamo sledeće iterativne korake.

   (k1) $\theta^* = \theta = 1$; $f_1(0) = (111)$;

   (k1.1) $\|f_1(0) \cap p_1\| = \|(111) \cap (100)\| = \|(100)\| = 1 \geq \theta^* \quad \Rightarrow \quad f_1(1) = f_1(0) \cap p_1 = (111) \cap (100) = (100)$;

   (k1.2) $\|f_1(1) \cap p_2\| = \|(100) \cap (101)\| = \|(100)\| = 1 \geq \theta^* \quad \Rightarrow \quad f_1(2) = f_1(1) \cap p_2 = (100) \cap (101) = (100)$;

   (k1.3) $\|f_1(2) \cap p_3\| = \|(100) \cap (001)\| = \|(000)\| = 0 < \theta^* \quad \Rightarrow \quad f_1(3) = f_1(2) = (100)$.

   Lako uočavamo da skup $B_1 = \{f_1(3)\}$ ne generiše skup S.

   (k2) $f_2(0) = (111)$;

   (k2.1) $\theta^* = \theta + \|f_1(3) \cap f_2(0) \cap p_1\| = 1 + \|(100) \cap (111) \cap (100)\| = 1 + \|(100)\| = 1 + 1 = 2$;
   $\|f_2(0) \cap p_1\| = \|(111) \cap (100)\| = \|(100)\| = 1 < 2 = \theta^* \quad \Rightarrow \quad f_2(1) = f_2(0) = (111)$;

   (k2.2) $\theta^* = \theta + \|f_1(3) \cap f_2(1) \cap p_2\| = 1 + \|(100) \cap (111) \cap (101)\| = 1 + \|(100)\| = 1 + 1 = 2$;
   $\|f_2(1) \cap p_2\| = \|(111) \cap (101)\| = \|(101)\| = 2 \geq 2 = \theta^* \quad \Rightarrow \quad f_2(2) = f_2(1) \cap p_2 = (111) \cap (101) = (101)$;

   (k2.3) $\theta^* = \theta + \|f_1(3) \cap f_2(2) \cap p_3\| = 1 + \|(100) \cap (101) \cap (001)\| = 1 + \|(000)\| = 1 + 0 = 1$;
   $\|f_2(2) \cap p_3\| = \|(101) \cap (001)\| = \|(001)\| = 1 \geq 1 = \theta^* \quad \Rightarrow \quad f_2(3) = f_2(2) \cap p_3 = (101) \cap (001) = (001)$.

   Skup $B_2 = \{f_1(3), f_2(3)\} = \{(100), (001)\}$ generiše skup S i njegova je baza jer je
   $p_1 = (100) = f_1(3) = (100)$,
   $p_2 = (101) = f_1(3) \cup f_2(3) = (100) \cup (001)$,
   $p_3 = (001) = f_2(3) = (001)$.

2. Posmatrajmo proizvoljne elemente $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ i $z = (z_1, \ldots, z_n)$ prostora $\mathbb{R}^n$.

   (d01) $d(x,y) = 0 \quad \Leftrightarrow \quad \max\limits_{i \in \{1,\ldots,n\}} |x_i - y_i| = 0,$
   $\Leftrightarrow \quad \forall i \in \{1, \ldots, n\},\ |x_i - y_i| = 0 \quad \Leftrightarrow \quad \forall i \in \{1, \ldots, n\},\ x_i = y_i \quad \Leftrightarrow \quad x = y.$

   (d02) $d(x,y) = \max\limits_{i \in \{1,\ldots,n\}} |x_i - y_i| = \max\limits_{i \in \{1,\ldots,n\}} |y_i - x_i| = d(y,x),$

   (d03) Za proizvoljne $a, b \in \mathbb{R}$ važi poznata nejednakost trougla $|a+b| \leq |a| + |b|$, te dobijamo
   $\forall i \in \{1, \ldots, n\},\ |x_i - z_i| = |(x_i - y_i) + (y_i - z_i)| \leq |x_i - y_i| + |y_i - z_i|$
   a odatle sledi
   $\max\limits_{i \in \{1,\ldots,n\}} |x_i - z_i| \leq \max\limits_{i \in \{1,\ldots,n\}} (|x_i - y_i| + |y_i - z_i|).$ [∗]
   Dokažimo da za proizvoljne n-torke a_i, $i \in \{1, \ldots, n\}$ i b_i, $i \in \{1, \ldots, n\}$ realnih brojeva važi
   $\max\limits_{i \in \{1,\ldots,n\}} (a_i + b_i) \leq \max\limits_{i \in \{1,\ldots,n\}} a_i + \max\limits_{i \in \{1,\ldots,n\}} b_i.$
   Neka je $\max\limits_{i \in \{1,\ldots,n\}} a_i = a_{i_0}$ za neko $i_0 \in \{1, \ldots, n\}$ i $\max\limits_{i \in \{1,\ldots,n\}} b_i = b_{j_0}$ za neko $j_0 \in \{1, \ldots, n\}$.
   Tada je $\forall i \in \{1, \ldots, n\},\ a_i \leq a_{i_0}$ i $\forall i \in \{1, \ldots, n\},\ b_i \leq b_{j_0}$. Sledi da je $\forall i \in \{1, \ldots, n\},\ a_i + b_i \leq a_{i_0} + b_{j_0}$, te je
   $\max\limits_{i \in \{1,\ldots,n\}} (a_i + b_i) \leq a_{i_0} + b_{j_0} = \max\limits_{i \in \{1,\ldots,n\}} a_i + \max\limits_{i \in \{1,\ldots,n\}} b_i.$
   Primenom ove nejednakosti u [∗] dobijamo
   $\max\limits_{i \in \{1,\ldots,n\}} |x_i - z_i| \leq \max\limits_{i \in \{1,\ldots,n\}} (|x_i - y_i| + |y_i - z_i|) \leq \max\limits_{i \in \{1,\ldots,n\}} |x_i - y_i| + \max\limits_{i \in \{1,\ldots,n\}} |y_i - z_i|,$
   dakle $d(x,z) \leq d(x,y) + d(y,z)$.

3. Tri rešavajuće funkcije vrše klasifikaciju na tri klase ω_1, ω_2 i ω_3.

   $\max\{d_1(A_1), d_2(A_1), d_3(A_1)\} = \max\{d_1(-2,1,1), d_2(-2,1,1), d_3(-2,1,1)\}$
   $= \max\{-4, -9, 1\} = 1 = d_3(A_1) \quad \Rightarrow \quad A_1 \in \omega_3,$

   $\max\{d_1(A_2), d_2(A_2), d_3(A_2)\} = \max\{d_1(8,3,3), d_2(8,3,3), d_3(8,3,3)\}$
   $= \max\{6, 11, -23\} = 11 = d_2(A_2) \quad \Rightarrow \quad A_2 \in \omega_2,$

   $\max\{d_1(A_3), d_2(A_3), d_3(A_3)\} = \max\{d_1(2,-5,-1), d_2(2,-5,-1), d_3(2,-5,-1)\}$
   $= \max\{30, 9, 9\} = 30 = d_1(A_3) \quad \Rightarrow \quad A_3 \in \omega_1,$

   $\max\{d_1(A_4), d_2(A_4), d_3(A_4)\} = \max\{d_1(2,2,2), d_2(2,2,2), d_3(2,2,2)\}$
   $= \max\{-1, -2, -9\} = -1 = d_1(A_4) \quad \Rightarrow \quad A_4 \in \omega_1,$

   $\max\{d_1(A_5), d_2(A_5), d_3(A_5)\} = \max\{d_1(-4,6,-2), d_2(-4,6,-2), d_3(-4,6,-2)\}$
   $= \max\{-25, 8, -13\} = 8 = d_2(A_5) \quad \Rightarrow \quad A_5 \in \omega_2,$

   $\max\{d_1(A_6), d_2(A_6), d_3(A_6)\} = \max\{d_1(1,-7,-5), d_2(1,-7,-5), d_3(1,-7,-5)\}$
   $= \max\{40, 28, 13\} = 40 = d_1(A_6) \quad \Rightarrow \quad A_6 \in \omega_1,$

$\max\{d_1(A_7), d_2(A_7), d_3(A_7)\} = \max\{d_1(-1,-2,-6), d_2(-1,-2,-6), d_3(-1,-2,-6)\}$
$= \max\{17, 33, 1\} = 33 = d_2(A_7) \quad \Rightarrow \quad A_7 \in \omega_2,$

$\max\{d_1(A_8), d_2(A_8), d_3(A_8)\} = \max\{d_1(8,-5,-2), d_2(8,-5,-2), d_3(8,-5,-2)\}$
$= \max\{43, 33, -4\} = 43 = d_1(A_8) \quad \Rightarrow \quad A_8 \in \omega_1.$

Klasifikacija je za rezultat dala klase

$\omega_1 = \{A_3, A_4, A_6, A_8\}, \quad \omega_2 = \{A_2, A_5, A_7\}, \quad \omega_3 = \{A_1\}.$

4. Svakoj od klasa ω_1, ω_2 i ω_3 pridružujemo redom odgovarajuće rešavajuće funkcije

$$d_1(x,y) = (x,y)\cdot z_1 - \frac{1}{2}|z_1|^2 = (x,y)\cdot(-6,2) - \frac{1}{2}|(-6,2)|^2 = -6x+2y-\frac{1}{2}\left((-6)^2+2^2\right) = -6x+2y-20,$$
$$d_2(x,y) = (x,y)\cdot z_2 - \frac{1}{2}|z_2|^2 = (x,y)\cdot(3,2) - \frac{1}{2}|(3,2)|^2 = 3x+2y-\frac{1}{2}\left(3^2+2^2\right) = 3x+2y-\frac{13}{2},$$
$$d_3(x,y) = (x,y)\cdot z_3 - \frac{1}{2}|z_3|^2 = (x,y)\cdot(2,-5) - \frac{1}{2}|(2,-5)|^2 = 2x-5y-\frac{1}{2}\left(3^2+2^2\right) = 2x-5y-\frac{29}{2}.$$

Njihovom primenom dobijamo sledeću klasifikaciju.

$\max\{d_1(A_1), d_2(A_1), d_3(A_1)\} = \max\{d_1(-2,1), d_2(-2,1), d_3(-2,1)\}$
$= \max\left\{-6, -\frac{21}{2}, -\frac{47}{2}\right\} = -6 = d_1(A_1) \quad \Rightarrow \quad A_1 \in \omega_1,$

$\max\{d_1(A_2), d_2(A_2), d_3(A_2)\} = \max\{d_1(8,3), d_2(8,3), d_3(8,3)\}$
$= \max\left\{-62, \frac{47}{2}, -\frac{27}{2}\right\} = \frac{47}{2} = d_2(A_2) \quad \Rightarrow \quad A_2 \in \omega_2,$

$\max\{d_1(A_3), d_2(A_3), d_3(A_3)\} = \max\{d_1(2,-5), d_2(2,-5), d_3(2,-5)\}$
$= \max\left\{-42, -\frac{21}{2}, \frac{29}{2}\right\} = \frac{29}{2} = d_3(A_3) \quad \Rightarrow \quad A_3 \in \omega_3,$

$\max\{d_1(A_4), d_2(A_4), d_3(A_4)\} = \max\{d_1(2,2), d_2(2,2), d_3(2,2)\}$
$= \max\left\{-28, \frac{7}{2}, -\frac{41}{2}\right\} = \frac{7}{2} = d_2(A_4) \quad \Rightarrow \quad A_4 \in \omega_2,$

$\max\{d_1(A_5), d_2(A_5), d_3(A_5)\} = \max\{d_1(-4,6), d_2(-4,6), d_3(-4,6)\}$
$= \max\left\{16, -\frac{13}{2}, -\frac{105}{2}\right\} = 16 = d_1(A_5) \quad \Rightarrow \quad A_5 \in \omega_1,$

$\max\{d_1(A_6), d_2(A_6), d_3(A_6)\} = \max\{d_1(1,-7), d_2(1,-7), d_3(1,-7)\}$
$= \max\left\{-40, -\frac{35}{2}, \frac{45}{2}\right\} = \frac{45}{2} = d_3(A_6) \quad \Rightarrow \quad A_6 \in \omega_3,$

$\max\{d_1(A_7), d_2(A_7), d_3(A_7)\} = \max\{d_1(-1,-2), d_2(-1,-2), d_3(-1,-2)\}$
$= \max\left\{-18, -\frac{27}{2}, -\frac{13}{2}\right\} = -\frac{13}{2} = d_3(A_7) \quad \Rightarrow \quad A_7 \in \omega_3,$

$\max\{d_1(A_8), d_2(A_8), d_3(A_8)\} = \max\{d_1(8,-5), d_2(8,-5), d_3(8,-5)\}$
$= \max\left\{-78, \frac{15}{2}, \frac{53}{2}\right\} = \frac{53}{2} = d_3(A_8) \quad \Rightarrow \quad A_8 \in \omega_3,$

Klasifikacija je za rezultat dala klase

$\omega_1 = \{A_1, A_5\}, \quad \omega_2 = \{A_2, A_4\}, \quad \omega_3 = \{A_3, A_6, A_7, A_8\}.$

Deobena prava klasa ω_1 i ω_2 je (vertikalna prava)

$H_{1,2} = \left\{(x,y) \in \mathbb{R}^3 \;\middle|\; d_1(x,y) = d_2(x,y)\right\} = \left\{(x,y) \in \mathbb{R}^3 \;\middle|\; -6x+2y-20 = 3x+2y-\frac{13}{2}\right\}$
$\quad = \left\{(x,y) \in \mathbb{R}^3 \;\middle|\; x = -\frac{3}{2}\right\}.$

Deobena prava klasa ω_1 i ω_3 je

$H_{1,3} = \left\{(x,y) \in \mathbb{R}^3 \;\middle|\; d_1(x,y) = d_3(x,y)\right\} = \left\{(x,y) \in \mathbb{R}^3 \;\middle|\; -6x+2y-20 = 2x-5y-\frac{29}{2}\right\}$
$\quad = \left\{(x,y) \in \mathbb{R}^3 \;\middle|\; y = \frac{8}{7}x+\frac{11}{14}\right\}.$

Deobena prava klasa ω_2 i ω_3 je

$H_{2,3} = \left\{(x,y) \in \mathbb{R}^3 \;\middle|\; d_2(x,y) = d_3(x,y)\right\} = \left\{(x,y) \in \mathbb{R}^3 \;\middle|\; 3x+2y-\frac{13}{2} = 2x-5y-\frac{29}{2}\right\}$
$\quad = \left\{(x,y) \in \mathbb{R}^3 \;\middle|\; 7y = -\frac{1}{7}x-\frac{8}{7}\right\}.$