1. Prirodni brojevi se klasifikuju u jednu od klasa ω_1, ω_2 i ω_3. Inicijalno klase ω_1, ω_2 i ω_3 sadrže sledeće prirodne brojeve:

   $\omega_1 = \{3,4,6\}, \quad \omega_2 = \{9\}, \quad \omega_3 = \{1,12\}.$

   Apriorne verovatnoće izbora klase za svrstavanje objekta su proporcionalne broju objekata u inicijalnom stanju klasa. Uslovne verovatnoće klasifikovanja objekta $n \in \mathbb{N}$ u klase ω_i, $i \in \{1,2,3\}$ su

   $$p(n \mid \omega_i) = 1 - \frac{[|n-i|]_3}{2}, \quad i \in \{1,2,3\},$$

   gde je $[m]_3$ ostatak pri delenju broja $m \in \mathbb{N} \cup \{0\}$ sa 3, odnosno

   $$[m]_3 = \begin{cases} 0 & , \ \exists k \in \mathbb{N} \cup \{0\}, \ m = 3k \\ 1 & , \ \exists k \in \mathbb{N} \cup \{0\}, \ m = 3k+1 \\ 2 & , \ \exists k \in \mathbb{N} \cup \{0\}, \ m = 3k+2 \end{cases} .$$

   Matrica gubitaka je $L = [L_{ij}]_{3\times 3}$ gde je $L_{ij} = |i-j|$, $i,j \in \{1,2,3\}$.

   Primenom uprošćenih uslovnih srednjih rizika (kao rešavajućih funkcija) klasifikovati broj 17 u jednu od klasa ω_1, ω_2, ω_3.

2. U prostoru $\mathbb{R}^2$ su date tačke

   $x_1 = (5,-2), \quad x_2 = (-1,-1), \quad x_3 = (-5,-4), \quad x_4 = (3,4),$
   $x_5 = (-1,4), \quad x_6 = (1,3), \quad x_7 = (-1,2), \quad x_8 = (3,5).$

   Algoritmom fiksnog praga klasifikovati tačke x_i, $i \in \{1,\ldots,8\}$ sa fiksnim pragom FP $= 6$, korišćenjem funkcije rastojanja $d((a_1,b_1),(a_2,b_2)) = |a_1 - a_2| + |b_1 - b_2|$.

3. U prostoru $\mathbb{R}^2$, algoritmom $K = 2$ unutrašnjih centara klasifikovati tačke

   $x_1 = (5,-2), \quad x_2 = (-5,-4), \quad x_3 = (3,4), \quad x_4 = (-1,4).$

   Pri klasifikovanju koristiti funkciju rastojanja

   $d((a_1,b_1),(a_2,b_2)) = |a_1 - a_2| + |b_1 - b_2|$

   i inicijalne centre klasa $z_1(1) = (-1,-1)$ i $z_2(1) = (3,1)$.

REŠENJA

1. Neka je k koeficijent proporcionalnosti kod apriornih verovatnoća izbora klasa. Iz $p(\omega_1) = k|\omega_1| = 3k$, $p(\omega_2) = k|\omega_2| = k$, $p(\omega_3) = k|\omega_3| = 2k$ i $p(\omega_1) + p(\omega_2) + p(\omega_3) = 3k + k + 2k = 6k = 1$ sledi $k = \frac{1}{6}$, te je

   $p(\omega_1) = \frac{3}{6}, \quad p(\omega_2) = \frac{1}{6}, \quad p(\omega_3) = \frac{2}{6}.$

   Matrica gubitaka je: $\quad L = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$

   Za $n = 17$, uslovne verovatnoće $p(17\,|\,\omega_j) = 1 - \frac{[|17-j|]_3}{2}$ klasifikovanja u klase ω_j, $j \in \{1,2,3\}$ su

   $p(17\,|\,\omega_1) = 1 - \frac{[|17-1|]_3}{2} = 1 - \frac{1}{2} = \frac{1}{2},$

   $p(17\,|\,\omega_2) = 1 - \frac{[|17-2|]_3}{2} = 1 - \frac{0}{2} = 1,$

   $p(17\,|\,\omega_3) = 1 - \frac{[|17-3|]_3}{2} = 1 - \frac{2}{2} = 0.$

   Uprošćeni uslovni srednji rizici

   $r_i(n) = \sum_{j=1}^{3} L_{ji}\, p(\omega_j) p(n\,|\,\omega_j) = \frac{3}{6} \cdot L_{1i} \cdot \frac{1}{2} + \frac{1}{6} \cdot L_{2i} \cdot 1 + \frac{2}{6} \cdot L_{3i} \cdot 0 = \frac{1}{4} \cdot L_{1i} + \frac{1}{6} \cdot L_{2i}$

   za $n = 17$ su redom

   $r_1(17) = \frac{1}{4} \cdot L_{11} + \frac{1}{6} \cdot L_{21} = \frac{1}{4} \cdot 0 + \frac{1}{6} \cdot 1 = \frac{1}{6},$

   $r_2(17) = \frac{1}{4} \cdot L_{12} + \frac{1}{6} \cdot L_{22} = \frac{1}{4} \cdot 1 + \frac{1}{6} \cdot 0 = \frac{1}{4},$

   $r_3(17) = \frac{1}{4} \cdot L_{13} + \frac{1}{6} \cdot L_{23} = \frac{1}{4} \cdot 2 + \frac{1}{6} \cdot 1 = \frac{4}{6}.$

   Sledi da $17 \in \omega_1$ jer je $r_1(17) = \min\{r_1(17), r_2(17), r_3(17)\}$.

2. Označimo redom sa S_i, $i = 1,2,\ldots$ i z_i, $i = 1,2,\ldots$ tekuće klase i njihove centre. Inicijalno je $S_i = \emptyset$, $i = 1,2,\ldots$

   (1) $S_1 := \emptyset \cup \{x_1\} := \{(5,-2)\}; \quad z_1 := x_1 := (5,-2); \quad bc := 1;$

   (2) $d(x_2,z_1) = d((-1,-1),(5,-2)) = |-1-5| + |-1-(-2)| = 7 > \text{FP};$
   sledi
   $S_2 := \emptyset \cup \{x_2\} := \{(-1,-1)\}; \quad z_2 := x_2 := (-1,-1); \quad bc := 2;$

   (3) $d(x_3,z_1) = d((-5,-4),(5,-2)) = |-5-5| + |-4-(-2)| = 12 > \text{FP};$
   $d(x_3,z_2) = d((-5,-4),(-1,-1)) = |-5-(-1)| + |-4-(-1)| = 7 > \text{FP};$
   sledi
   $S_3 := \emptyset \cup \{x_3\} := \{(-5,-4)\}; \quad z_3 := x_3 := (-5,-4); \quad bc := 3;$

   (4) $d(x_4,z_1) = d((3,4),(5,-2)) = |3-5| + |4-(-2)| = 8 > \text{FP};$
   $d(x_4,z_2) = d((3,4),(-1,-1)) = |3-(-1)| + |4-(-1)| = 9 > \text{FP};$
   $d(x_4,z_3) = d((3,4),(-5,-4)) = |3-(-5)| + |4-(-4)| = 16 > \text{FP};$
   sledi
   $S_4 := \emptyset \cup \{x_4\} := \{(3,4)\}; \quad z_4 := x_4 := (3,4); \quad bc := 4;$

   (5) $d(x_5,z_1) = d((-1,4),(5,-2)) = |-1-5| + |4-(-2)| = 12 > \text{FP};$
   $d(x_5,z_2) = d((-1,4),(-1,-1)) = |-1-(-1)| + |4-(-1)| = 5 \leq \text{FP};$
   $d(x_5,z_3) = d((-1,4),(-5,-4)) = |-1-(-5)| + |4-(-4)| = 12 > \text{FP};$
   $d(x_5,z_4) = d((-1,4),(3,4)) = |-1-3| + |4-4| = 4 \leq \text{FP};$
   pri tome je
   $\min_{i \in \{1,\ldots,4\}} d(x_5,z_i) = d(x_5,z_4) = 4$
   te sledi
   $S_4 := S_4 \cup \{x_5\} := \{x_4,x_5\} := \{(3,4),(-1,4)\};$

   (6) $d(x_6,z_1) = d((1,3),(5,-2)) = |1-5| + |3-(-2)| = 6 \leq \text{FP};$
   $d(x_6,z_2) = d((1,3),(-1,-1)) = |1-(-1)| + |3-(-1)| = 6 \leq \text{FP};$
   $d(x_6,z_3) = d((1,3),(-5,-4)) = |1-(-5)| + |3-(-4)| = 13 > \text{FP};$
   $d(x_6,z_4) = d((1,3),(3,4)) = |1-3| + |3-4| = 3 \leq \text{FP};$
   pri tome je
   $\min_{i \in \{1,\ldots,4\}} d(x_6,z_i) = d(x_6,z_4) = 3$
   te sledi
   $S_4 := S_4 \cup \{x_6\} := \{x_4,x_5,x_6\} := \{(3,4),(-1,4),(1,3)\};$

(7) $d(x_7,z_1) = d((-1,2),(5,-2)) = |-1-5| + |2-(-2)| = 10 > \text{FP};$
$d(x_7,z_2) = d((-1,2),(-1,-1)) = |-1-(-1)| + |2-(-1)| = 3 \le \text{FP};$
$d(x_7,z_3) = d((-1,2),(-5,-4)) = |-1-(-5)| + |2-(-4)| = 10 > \text{FP};$
$d(x_7,z_4) = d((-1,2),(3,4)) = |-1-3| + |2-4| = 6 \le \text{FP};$
pri tome je
$$\min_{i\in\{1,\dots,4\}} d(x_7,z_i) = d(x_7,z_2) = 3$$
te sledi
$S_2{:=}S_2 \cup \{x_7\}{:=}\{x_2,x_7\}{:=}\{(-1,-1),(-1,2)\};$

(8) $d(x_8,z_1) = d((3,5),(5,-2)) = |3-5| + |5-(-2)| = 9 > \text{FP};$
$d(x_8,z_2) = d((3,5),(-1,-1)) = |3-(-1)| + |5-(-1)| = 10 > \text{FP};$
$d(x_8,z_3) = d((3,5),(-5,-4)) = |3-(-5)| + |5-(-4)| = 17 > \text{FP};$
$d(x_8,z_4) = d((3,5),(3,4)) = |3-3| + |5-4| = 1 \le \text{FP};$
te sledi
$S_4{:=}S_4 \cup \{x_8\}{:=}\{x_4,x_5,x_6,x_8\}{:=}\{(3,4),(-1,4),(1,3),(3,5)\};$

Dakle, dobijeni su sledeći klasteri:

$S_1 = \{x_1\}; \quad S_2 = \{x_2,x_7\}; \quad S_3 = \{x_3\}; \quad S_4 = \{x_4,x_5,x_6,x_8\}.$

3. Neka su $S_1(i)$, $i \in \mathbb{N}$ i $S_2(i)$, $i \in \mathbb{N}$ tekuće klase, a $z_1(i)$, $i \in \mathbb{N}$ i $z_2(i)$, $i \in \mathbb{N}$ tekući centri klasa u i-toj iteraciji.

[I1] $S_1(1) = S_2(1) = \emptyset.$

$d(x_1,z_1(1)) = d((5,-2),(-1,-1)) = |5-(-1)| + |-2-(-1)| = 7,$
$d(x_1,z_2(1)) = d((5,-2),(3,1)) = |5-3| + |-2-1| = 5,$
$\Rightarrow \quad S_2(1) = S_2(1) \cup \{x_1\}.$

$d(x_2,z_1(1)) = d((-5,-4),(-1,-1)) = |-5-(-1)| + |-4-(-1)| = 7,$
$d(x_2,z_2(1)) = d((-5,-4),(3,1)) = |-5-3| + |-4-1| = 13,$
$\Rightarrow \quad S_1(1) = S_1(1) \cup \{x_2\}.$

$d(x_3,z_1(1)) = d((3,4),(-1,-1)) = |3-(-1)| + |4-(-1)| = 9,$
$d(x_3,z_2(1)) = d((3,4),(3,1)) = |3-3| + |4-1| = 3,$
$\Rightarrow \quad S_2(1) = S_2(1) \cup \{x_3\}.$

$d(x_4,z_1(1)) = d((-1,4),(-1,-1)) = |-1-(-1)| + |4-(-1)| = 5,$
$d(x_4,z_2(1)) = d((-1,4),(3,1)) = |-1-3| + |4-1| = 7,$
$\Rightarrow \quad S_1(1) = S_1(1) \cup \{x_4\}.$

Dakle, $S_1(1) = \{x_2,x_4\}$ i $S_2(1) = \{x_1,x_3\}$. Novi centri klasa su
$$z_1(2) = \frac{1}{|S_1(1)|}(x_2+x_4) = \frac{1}{2}((-5,-4)+(-1,4)) = (-3,0),$$
$$z_2(2) = \frac{1}{|S_2(1)|}(x_1+x_3) = \frac{1}{2}((5,-2)+(3,4)) = (4,1).$$

[I2] $S_1(2) = S_2(2) = \emptyset.$

$d(x_1,z_1(2)) = d((5,-2),(-3,0)) = |5-(-3)| + |-2-0| = 10,$
$d(x_1,z_2(2)) = d((5,-2),(4,1)) = |5-4| + |-2-1| = 4,$
$\Rightarrow \quad S_2(2) = S_2(2) \cup \{x_1\}.$

$d(x_2,z_1(2)) = d((-5,-4),(-3,0)) = |-5-(-3)| + |-4-0| = 6,$
$d(x_2,z_2(2)) = d((-5,-4),(4,1)) = |-5-4| + |-4-1| = 14,$
$\Rightarrow \quad S_1(2) = S_1(2) \cup \{x_2\}.$

$d(x_3,z_1(2)) = d((3,4),(-3,0)) = |3-(-3)| + |4-0| = 10,$
$d(x_3,z_2(2)) = d((3,4),(4,1)) = |3-4| + |4-1| = 4,$
$\Rightarrow \quad S_2(2) = S_2(2) \cup \{x_3\}.$

$d(x_4,z_1(2)) = d((-1,4),(-3,0)) = |-1-(-3)| + |4-0| = 6,$
$d(x_4,z_2(2)) = d((-1,4),(4,1)) = |-1-4| + |4-1| = 8,$
$\Rightarrow \quad S_1(2) = S_1(2) \cup \{x_4\}.$

Dakle, $S_1(2) = \{x_2,x_4\}$ i $S_2(2) = \{x_1,x_3\}$. Novi centri klasa su
$$z_1(3) = \frac{1}{|S_1(2)|}(x_2+x_4) = \frac{1}{2}((-5,-4)+(-1,4)) = (-3,0),$$
$$z_2(3) = \frac{1}{|S_2(2)|}(x_1+x_3) = \frac{1}{2}((5,-2)+(3,4)) = (4,1).$$
Kako je $z_1(3) = z_1(2)$ i $z_2(3) = z_2(2)$, postupak je završen, i
$S_1 = S_1(2) = \{x_2,x_4\}, \quad S_2 = S_2(2) = \{x_1,x_3\}$
je konačna klasifikacija.