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MINIMUM, MAXIMUM AND MEANS AGGREGATION OF DISTANCE FUNCTIONS

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Abstract. In this paper, some features of the aggregation operators min, max and *generalized means*, that are relevant for the construction of new distance functions by applying aggregation operator on sequence of some given distance functions, are examined and presented. The features of the constructed distance function depend on the characteristics of the applied aggregation operator.

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1. Introduction

Distance functions and metrics have significant role in many scientific disciplines and applications, see [2, 3, 6]. For example, in automatic image segmentation distance functions represent a decision criterion for classifying pixels into segments, see [6]. In decision making process, distance function can be a criterion for decision evaluation. In mathematical models and applications where decision making process is based on multiple criteria, aggregation functions are natural and frequently used way to join them into one decisive criterion, see [1, 4, 5]. By choosing an appropriate aggregation function, we can model a information fusion criterion in accordance with the intuitive perception.

In this paper, we consider construction of new distance function by applying aggregation operators min, max and *generalized means* on some given distance functions. In paper [6], it is shown that by applying aggregation operator on distance functions, new distance function is constructed. The features of the constructed distance function depend both on the characteristics of the applied aggregation function and on the initial distance functions. In the section 2 we present related definitions and previous results about aggregated distance functions. In the section 3 we present previous and some new results about relevant properties of min, max and *generalized means* aggregation operators.

2. Preliminaries and previous research

Aggregation operators are type of fuzzy operations, which have found large application in various engineering disciplines, see [4, 5].

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Definition 2.1 (Aggregation function)**.** For $n \in \mathbb{N}$, an n-ary *aggregation function* is a function $A : [0,1]^n \to [0,1]$ with the following properties.

(a01) *Boundary conditions* $A(0,\dots,0) = 0$ and $A(1,\dots,1) = 1$ hold.

(a02) A function A is monotonically non-decreasing in each component, i.e.,

$$\forall i \in \{1,\dots,n\}\,,\ a_i \le b_i \quad\Rightarrow\quad A(a_1,\dots,a_n) \le A(b_1,\dots,b_n)$$

hold for each $(a_1,\dots,a_n) \in [0,1]^n$ and $(b_1,\dots,b_n) \in [0,1]^n$.

Additionaly, in case $n = 1$, $A(x) = x$ for all $x \in [0,1]$.

Definition 2.2 (Extended aggregation function)**.** An *extended aggregation function* (*aggregation operator*) is a mapping $A : \bigcup_{n=1}^{\infty} [0,1]^n \to [0,1]$ such that every restriction $A_{[n]} : [0,1]^n \to [0,1]$ is an n-ary aggregation function.

In addition, function A could have some of the following additional properties, which are of interest for construction of new distance functions by applying aggregation operators on sequence of initial distance functions. With $A_{[n]}$ we denote aggregation function which is restriction of A on $[0,1]^n$.

(a03) Every restriction $A_{[n]}$ is continuous.

(a04) Function A is symmetric, i.e., for each $n \in \mathbb{N}$, each $(a_1,\dots,a_n) \in [0,1]^n$ and each permutation p of the set $\{1,\dots,n\}$ hold
$A_{[n]}(a_1,\dots,a_n) = A_{[n]}(a_{p(1)},\dots,a_{p(n)})$.

(a05) Function A is idempotent, i.e., for each $n \in \mathbb{N}$, and all $(a,\dots,a) \in [0,1]^n$ hold $A_{[n]}(a,\dots,a) = a$.

(a06) Function A is additive, i.e., for each $n \in \mathbb{N}$ and all $(a_1,\dots,a_n) \in [0,1]^n$, $(b_1,\dots,b_n) \in [0,1]^n$ that satisfy $(a_1+b_1,\dots,a_n+b_n) \in [0,1]^n$ hold
$A_{[n]}(a_1+b_1,\dots,a_n+b_n) = A_{[n]}(a_1,\dots,a_n) + A_{[n]}(b_1,\dots,b_n)$.

(a07) Function A is sub-additive, i.e., for each $n \in \mathbb{N}$ and $(a_1,\dots,a_n) \in [0,1]^n$, $(b_1,\dots,b_n) \in [0,1]^n$ that satisfy $(a_1+b_1,\dots,a_n+b_n) \in [0,1]^n$ hold
$A_{[n]}(a_1+b_1,\dots,a_n+b_n) \le A_{[n]}(a_1,\dots,a_n) + A_{[n]}(b_1,\dots,b_n)$.

(a08) Function A is positively homogeneous, i.e., for each $t \ge 0$, each $n \in \mathbb{N}$ and all $(a_1,\dots,a_n) \in [0,1]^n$ that satisfy $(ta_1,\dots,ta_n) \in [0,1]^n$ hold
$A_{[n]}(ta_1,\dots,ta_n) = tA_{[n]}(a_1,\dots,a_n)$.

(a09) Function A is positively sub-homogeneous, i.e., for each $t \ge 0$, each $n \in \mathbb{N}$ and all $(a_1,\dots,a_n) \in [0,1]^n$ that satisfy $(ta_1,\dots,ta_n) \in [0,1]^n$ hold
$A_{[n]}(ta_1,\dots,ta_n) \le tA_{[n]}(a_1,\dots,a_n)$.

(a10) For each $n \in \mathbb{N}$, holds: $A_{[n]}(a_1,\dots,a_n) = 0 \ \Rightarrow\ \forall i \in \{1,\dots,n\}\,,\ a_i = 0$.

(a11) For each $n \in \mathbb{N}$ holds: $A_{[n]}(a_1,\dots,a_n) = 0 \ \Rightarrow\ \exists i \in \{1,\dots,n\}\,,\ a_i = 0$.

Distance functions could be interpreted as a measure of difference between two objects. They have many applications in diverse areas of natural and social sciences, see [2, 3]. In [3], detailed review of various type of distance functions and their possible properties is given, as well as a presentation of many aspects of their application in numerous mathematic and other disciplines.

Definition 2.3. Let $X \neq \emptyset$ be an arbitrary set. A *distance function* on set (space) X is a function $d : X^2 \to [0, \infty)$, which have the following properties.
(d01) $\forall x, y \in X,\ \ d(x, y) = d(y, x),$ (*symmetry*)
(d02) $\forall x \in X,\ \ d(x, x) = 0.$ (*reflexivity*)
Ordered pair (X, d) is then called *a space with a distance.*

Definition 2.4. In space $X \neq \emptyset$, function $d : X^2 \to [0, \infty)$ could have following important properties.
(d03) $\forall x, y \in X,\ \ d(x, y) = 0 \ \Rightarrow\ x = y$ (*identity of indiscernibles*).
(d04) $\forall x, y, z \in X,\ \ d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*).
(d05) For certain constant $C \in [1, \infty)$ applies:
$\forall x, y, z \in X,\ \ d(x, z) \leq C\,(d(x, y) + d(y, z))$ (*C-triangle inequality*).
(d06) $d : X^2 \to [0, 1] \ \ \vee\ \ \exists a > 0,\ d : X^2 \to [0, a]$ (*boundedness*).

A distance function $d : X^2 \to [0, \infty)$ is a *metric* on a set X if satisfies identity of indiscernibles (d03) and triangle inequality (d04).

In the paper [6], method for the construction of new distance functions and metrics by applying aggregation operators on sequence of initial distance functions and metrics, is presented. Let $d_i : X^2 \to [0, 1]$, $i \in \mathbb{N}$ be bounded distance functions on set $X \neq \emptyset$, and $A : \bigcup_{n=2}^{\infty} [0, 1]^n \to [0, 1]$ be an arbitrary aggregation operator. Let function $d : X^2 \times \mathbb{N} \setminus \{1\} \to [0, 1]$ be defined with

$$d(x, y; n) = A\,(d_1(x, y), \ldots, d_n(x, y)), \quad x, y \in X, \quad n \in \mathbb{N} \setminus \{1\},$$

and for each $n \geq 2$ function $d_{[n]} : X^2 \to [0, 1]$ be defined with

$$d_{[n]}(x, y) = d(x, y; n) = A\,(d_1(x, y), \ldots, d_n(x, y)), \quad x, y \in X.$$

In the following theorem (see [6]), certain properties of function $d_{[n]}$ are stated, depending on properties of distance function d_i and properties of aggregation operator A, primarily (a07), (a09), (a10) and (a11) from the page 2.

Theorem 2.5. *Let $d_i : X^2 \to [0, 1]$, $i \in \mathbb{N}$ be arbitrary sequence of distance functions, and let $A : \bigcup_{n=1}^{\infty} [0, 1]^n \to [0, 1]$ be an arbitrary extended aggregation function. Then for each $n \in \mathbb{N}$, and for all functions $d_{[n]} : X^2 \to [0, 1]$ defined with $d_{[n]}(x, y) = A\,(d_1(x, y), \ldots, d_n(x, y))$, $x, y \in X$ following statements hold.*

(ad01) *Function $d_{[n]}$ is distance function.*

(ad02) *If each of distance function d_i, $i \in \mathbb{N}$ satisfies identity of indiscernibles* (d03) *and operator A has property* (a11) *for each $n \in \mathbb{N}$, then for function $d_{[n]}$ identity of indiscernibles* (d03) *holds.*

(ad03) *If for at least one distance function d_i, $i \in \{1, \ldots, n\}$ applies identity of indiscernibles* (d03) *and operator A has property* (a10) *for $n \in \mathbb{N}$, then for function $d_{[n]}$ identity of indiscernibles* (d03) *holds.*

(ad04) *Let all d_i, $i \in \mathbb{N}$ be metric, and let A be sub-additive function (property* (a07)*) which restriction on set $\bigcup_{n=1}^{\infty} [0, 1]^n$ is an aggregation operator. If the aggregation operator $A : \bigcup_{n=1}^{\infty} [0, 1]^n \to [0, 1]$ has property* (a11) *for each $n \in \mathbb{N}$, then function $d_{[n]}$ is metric for each $n \in \mathbb{N}$.*

(ad05) *Let for each function* d_i, $i \in \mathbb{N}$ *C-triangle inequality* (d05) *holds. Let* $A : \bigcup_{n=1}^{\infty} [0,\infty)^n \to [0,\infty)$ *be function which restriction on the set* $\bigcup_{n=1}^{\infty} [0,1]^n$ *is aggregation operator that is sub-additive and positively sub-homogeneous, i.e. has properties* (a07) *and* (a09). *Then for functions* $d_{[n]}$, $n \in \mathbb{N}$ *C-triangle inequality* (d05) *holds. Besides that, if for each function* d_i, $i \in \mathbb{N}$ *C-triangle inequality applies with corresponding constant* C_i, $i \in \mathbb{N}$, *then C-triangle inequality for functions* $d_{[n]}$, $n \in \mathbb{N}$ *holds with corresponding constants* $C_{[n]} = \max\{C_1, \ldots, C_n\}$, $n \geq 2$.

(ad06) *Let* A *be continuous aggregation operator. If space* X *is equipped with topological structure and each of distance function* d_i, $i \in \mathbb{N}$ *is continuous on* X^2, *then distance function* $d_{[n]}$ *is continuous on* X^2 *for* $n \in \mathbb{N}$.

The distance functions constructed by the given procedure can find various applications. E.g., in [6], the application in image segmentation is shown.

3. Properties of min, max and generalized means operators

In the paper [6], extended aggregation functions min, max, *extended weighted arithmetic mean* and *extended generalized means* (see [1, 5]) and their properties (a07), (a09), (a10) and (a11) are examined, from the aspect of their applications in the Theorem 2.5. It is known, see [1, 5], which of the properties (a03), (a04) and (a05), these functions satisfy or not. In the following examples we further discuss other properties of mentioned aggregation operators.

Example 3.1 (Extended WAM)**.** For arbitrary family of real numbers

$$\omega = \left\{ \omega_{n,i} \geq 0 \,\middle|\, n \in \mathbb{N},\ i \in \{1, ..., n\}\,,\ \sum_{i=1}^{n} \omega_{n,i} = 1 \right\},$$

with the following equation a continuous and idempotent, but not symmetrical aggregation operator $\mathsf{WAM}_\omega : \bigcup_{n=1}^{\infty} [0,1]^n \to [0,1]$ is defined,

$$\mathsf{WAM}_\omega(a_1, \ldots, a_n) = \omega_{n,1} a_1 + \cdots + \omega_{n,n} a_n, \quad (a_1, \ldots, a_n) \in [0,1]^n. \tag{3.1}$$

Operator WAM_ω is called *extended weighted arithmetic mean*, see [5]. For each $n \in \mathbb{N}$, restriction $\mathsf{WAM}_\omega^{[n]} : [0,1]^n \to [0,1]$ of operator WAM_ω on the $[0,1]^n$ is a restriction of one linear transformation from a space $\mathbb{R}^n$ to $\mathbb{R}$. This means that function WAM_ω is additive and positively homogeneous, and so has the properties (a06), (a07), (a08) and (a09). Also, it obviously satisfies (a11), and the property (a10) has only if $\omega_{n,i} > 0$ for all $n \in \mathbb{N}$, $i \in \{1, \ldots, n\}$. ✓

Example 3.2 (min and max)**.** Functions $A_{\min}(a_1, \ldots, a_n) = \min(a_1, \ldots, a_n)$, $A_{\max}(a_1, \ldots, a_n) = \max(a_1, \ldots, a_n)$, are obviously continuous, symmetrical and idempotent aggregation functions. Let us prove that min is positively homogeneous, and for max proof can be performed in the same way. For $n \in \mathbb{N}$, let $(a_1, \ldots, a_n) \in [0,1]^n$ and $t \geq 0$ satisfy $(ta_1, \ldots, ta_n) \in [0,1]^n$, and let for some $p \in \{1, \dots, n\}$, $\min(a_1, \dots, a_n) = a_p$. From previous, it is obvious that $a_p \le a_i$ for all $i \in \{1, \dots, n\}$. Because $t \ge 0$, it can be concluded that $ta_p \le ta_i$, $i \in \{1, \dots, n\}$, which results $\min(ta_1, \dots, ta_n) = ta_p = t\min(a_1, \dots, a_n)$. ✓

Example 3.3 (Extended generalized means)**.** For $\alpha \neq 0$ with

$$GA_\alpha(a_1, \dots, a_n) = \left(\frac{a_1^\alpha + \cdots + a_n^\alpha}{n}\right)^{\frac{1}{\alpha}}, \quad \alpha > 0$$

i.e.,

$$GA_\alpha(a_1, \dots, a_n) = \begin{cases} \left(\frac{a_1^\alpha + \cdots + a_n^\alpha}{n}\right)^{\frac{1}{\alpha}} & , \ \forall i,\ a_i > 0 \\ 0 & , \ \exists i,\ a_i = 0 \end{cases}, \quad \alpha < 0.$$

is defined a continuous, symmetric and idempotent aggregation function, which is also called *extended generalized average functions.* For some special values α, we obtain the well known functions *arithmetic mean* MA, *geometric mean* MG and *harmonic mean* MH. In [6], properties (a07), (a09), (a10) and (a11) of MA and MG were examined. Here, we will further consider the MH operator.

[MA] For $\alpha = 1$ the operator is called *extended arithmetic mean*,

$$\mathsf{MA}(a_1, \dots, a_n) = \frac{1}{n}(a_1 + \cdots + a_n).$$

As a linear transformation, MA is additive, positively homogeneous, positively subhomogeneous, subadditive and satisfies (a10) and (a11).

[MG] If $\alpha \to 0$ the operator GA_α converges to extended geometric mean

$$\mathsf{MG}(a_1, \dots, a_n) = (a_1 \cdot \ldots \cdot a_n)^{\frac{1}{n}}.$$

Function MG is positively homogeneous and positively subhomogeneous, because for all $t \ge 0$ and $(a_1, \dots, a_n) \in [0,1]^n$, $n \ge 2$ hold

$$\begin{aligned} \mathsf{MG}(ta_1, \dots, ta_n) &= ((ta_1) \cdot \dots \cdot (ta_n))^{\frac{1}{n}} \\ &= t\,(a_1 \cdot \dots \cdot a_n)^{\frac{1}{n}} = t\mathsf{MG}(a_1, \dots, a_n). \end{aligned}$$

Function MG is not subadditive (see [6]), it obviously have the property (a11), but do not have the property (a10).

[MH] By choosing $\alpha = -1$, extended harmonic mean is obtained,

$$\mathsf{MH}(a_1, \dots, a_n) = \begin{cases} \dfrac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} & , \quad \forall i,\ a_i \neq 0 \\ 0 & , \quad \exists i,\ a_i = 0 \end{cases}.$$

Harmonic mean does not satisfy the property (a07), neither (a06), because e.g., for $n = 2$

$$\mathsf{MH}(0.5 + 0.1, 0.25 + 0.2) = \frac{18}{35} > \mathsf{MH}(0.5, 0.25) + \mathsf{MH}(0.1, 0.2) = \frac{7}{15}.$$

This function is positively homogeneous and positively subhomogeneous, which follows from the equations below. If $\exists i$, such that $a_i = 0$, then $\mathsf{MH}(ta_1, \dots, ta_n) = t\mathsf{MH}(a_1, \dots, a_n) = 0$, otherwise

$$\begin{aligned} \mathsf{MH}(ta_1, \dots, ta_n) &= \frac{n}{\frac{1}{ta_1} + \cdots + \frac{1}{ta_n}} = \frac{n}{\frac{1}{t}\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right)} \\ &= \frac{tn}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} = t\mathsf{MH}(a_1, \dots, a_n). \end{aligned}$$

From the definition of MH function, follows that it satisfy (a11), and not satisfy (a10).

In the general case, function GA_α, $\alpha \in \mathbb{R}\backslash\{0\}$, is continuous, symmetric and idempotent (properties (a03), (a04) and (a05)) that follows from its definition. It is also positively homogeneous and positively subhomogeneous (properties (a08) and (a09)) for each $\alpha \in \mathbb{R}\backslash\{0\}$, because for all $n \in \mathbb{N}$, $(a_1, \ldots, a_n) \in [0,1]^n$ and $t \geq 0$ such that $(ta_1, \ldots, ta_n) \in [0,1]^n$ the following is obtained.

- For $\alpha < 0$, if $\exists i \in \{1, \ldots, n\}$ such that $a_i = 0$, then $ta_i = 0$ and
  $GA_\alpha(a_1, \ldots, a_n) = GA_\alpha(ta_1, \ldots, ta_n) = 0$,
  so obviously $GA_\alpha(ta_1, \ldots, ta_n) = t\, GA_\alpha(a_1, \ldots, a_n)$.
- For all $\alpha > 0$, or in a case $\alpha < 0$ and $\forall i \in \{1, ..., n\}$, $a_i > 0$ follows
  $$GA_\alpha(ta_1, \ldots, ta_n) = \left(\frac{(ta_1)^\alpha + \cdots + (ta_n)^\alpha}{n}\right)^{\frac{1}{\alpha}}$$
  $$= \left(t^\alpha \frac{a_1^\alpha + \cdots + a_n^\alpha}{n}\right)^{\frac{1}{\alpha}} = t\left(\frac{a_1^\alpha + \cdots + a_n^\alpha}{n}\right)^{\frac{1}{\alpha}} = t\, GA_\alpha(a_1, \ldots, a_n). \quad \checkmark$$

4. Conclusions

We consider this method of constructing of the distance functions can give good results in the procedures for removing the noise in the image by choosing an appropriate aggregation operators, initial distance functions and other suitable parameters (such as the appropriate pixel-descriptor).

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References

[1] Calvo T., Mesiar R., Generalized medians, Fuzzy Sets and Systems 124:1 (2001), 59-64.

[2] Ćurić V., Distance functions and their use in adaptive mathematical morphology, Ph. D. thesis, Uppsala, 2014.

[3] Deza M. M., Deza E., Encyclopedia of Distances, Springer-Verlag, 2012.

[4] Dubois D., Prade H. On the use of aggregation operations in information fusion processes, Fuzzy Sets and Systems 142:1 (2004), 143-161.

[5] Grabish, M., Marichal, J.-L., Mesiar, R., Pap, E., Aggregation Functions, Cambridge University Press, 2009.

[6] Nedović, L., Ralević, N., Pavkov, I., Aggregated distance functions and their application in the fuzzy c-means algorithm, Soft Computing 22 (2018), 4723-4739.