Absolutely Monotone Real Set Functions

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Abstract—We present a class of absolutely monotone and signed stable set functions with \( m(\emptyset) = 0 \), AMSS. The representation of a set function from AMSS as a symmetric maximum of two monotone set function is obtained. We present three integrals of a real-valued measurable function based on \( m \in \text{AMSS} \).

Keywords: symmetric maximum, absolutely monotone set function, sign stable set function

I. INTRODUCTION

A generalization of the classical probability measure, a fuzzy measure (capacity), together with fuzzy integrals, has many applications in economics, pattern recognition, and decision analysis [3], [4], [5], [20], [21]. It is proven in [1], [16] that a real-valued set function \( m \), \( m(\emptyset) \), belongs to the space of set functions with the bounded chain variation, \( BV \), if it can be represented as a difference of two finite fuzzy measures, vanishing at the empty set, i.e. \( m = \nu_1 - \nu_2 \).

The most important integral defined with respect to a fuzzy measure is introduced by G. Choquet in [2]. The Choquet integral of a non-negative real-valued function \( f \) based on a fuzzy measure \( m : \mathcal{A} \to [0, \infty] \) is defined by

\[
C_m(f) = \int_0^\infty m(\{x | f(x) \geq t\})dt. \tag{1}
\]

The Choquet integral is defined with respect to non-monotonic, real-valued set functions with the bounded chain variation [14], and it is also known as the asymmetric Choquet integral [3], [14], [16]. The asymmetric Choquet integral is linear with respect to \( m \).

An another important integral based on a fuzzy measure \( m : \mathcal{A} \to [0, 1] \), introduced in [18] by M. Sugeno, is the Sugeno integral defined by

\[
S_m(f) = \sup_{t \in [0,1]} \{ t \wedge m(\{x | f(x) \geq t\}) \}. \tag{2}
\]

The symmetric Sugeno integral of a real-valued function, introduced in [5], is also defined with respect to a fuzzy measure.

The Choquet-like integral related to some non-decreasing function \( g : [0, 1] \to [0, \infty] \), \( g(0) = 0 \), defined for a non-negative function and a fuzzy measure \( m \) is given by

\[
C^g_m(f) = g^{-1}(C_{g\circ m}(g \circ f)). \tag{3}
\]

This integral is introduced in [9] and it is also defined for a real-valued function \( f \), if for \( g \) is taken its odd extension on the real line [9].

In [13] the authors introduced an absolutely monotone and sign stable set function \( m : \mathcal{A} \to [-1, 1] \), \( m(\emptyset) = 0 \). It is shown that \( m \) can be represented by a pseudo-difference of two fuzzy measures. The class of such set functions is denoted by AMSS.

The aim of this paper is to present the class AMSS, and to propose different types of monotonic integrals based on a set function from AMSS. Our previous papers [12], [13] are devoted to non-monotonic integrals.

The paper is organized as follows. In Section 2 the preliminary notions and definitions are given. In Section 3 we consider the class of absolutely monotone and sign stable set functions, AMSS. In Section 4 we propose definitions of three integrals based on \( m \in \text{AMSS} \), \( m(X) < 1 \) and present their basic properties.

II. PRELIMINARIES

The symmetric maximum is originally introduced in [5], [6].

**Definition 1:** The symmetric maximum \( \boxplus : [-a, a]^2 \to [-a, a] \), \( a \in \mathbb{R}^+ \) is given by

\[
x \boxplus y := \text{sign}(x+y)(|x| \vee |y|).
\]

The symmetric maximum is a commutative, non-decreasing operation with neutral element 0 and annihilator \( a \). It is not associative, nor continuous.

Figure 1. Symmetric maximum
Let $I$ be a set of elements from $[-a,a]$. The symmetric maximum is defined by:

$$\bigoplus_{i \in I} x_i = \left( \bigoplus_{x_i \geq 0} x_i \right) \oplus \left( \bigoplus_{x_i < 0} x_i \right) = \sup_{x_i \geq 0} x_i \oplus \inf_{x_i < 0} x_i.$$ 

Let us assume $X$ is a non-empty universal set. Let $\mathcal{A}$ be a $\sigma$-algebra of subsets of $X$.

**Definition 2**: A set function, $m : \mathcal{A} \to [-a,a]$, $a \in \mathbb{R}^+$, $m(\varnothing) = 0$ is

(i) **non-negative** if $m(A) \geq 0$ for all $A \in \mathcal{A}$,

(ii) **revised monotone** if for all $A, B \in \mathcal{A}$, $A \cap B = \varnothing$, we have

   a) $m(A) \geq 0, m(B) \geq 0, m(A) \vee m(B) > 0 \Rightarrow m(A \cup B) \geq m(A) \vee m(B)$;

   b) $m(A) \leq 0, m(B) \leq 0, m(A) \wedge m(B) < 0 \Rightarrow m(A \cup B) \leq m(A) \wedge m(B)$;

   c) $m(A) > 0, m(B) < 0 \Rightarrow m(B) \leq m(A \cup B) \leq m(A)$.

(iii) **fuzzy measure** if it is non-negative and monotone, i.e. if for all $A, B \in \mathcal{A}$, $A \subset B$ we have $|m(A)| \leq |m(B)|$.

(iv) **signed fuzzy measure** if it is revised monotone.

We introduce the notion of absolutely monotone set functions.

**Definition 3**: A set function $m : \mathcal{A} \to \mathbb{R}$ is

(i) **absolutely monotone** if for all $A, B \in \mathcal{A}$, $A \subset B$ we have $|m(A)| \leq |m(B)|$.

(ii) **absolutely strictly monotone** if for all $A, B \in \mathcal{A}$, $A \subset B$, $A \neq B$, we have $|m(A)| < |m(B)|$.

Obviously, any fuzzy measure is an absolutely monotone set function.

**Example 1**: Let $X = A \cup B$, $A \cap B = \varnothing$, such that $\text{card}(A) = \text{card}(B) = n$. Let $m$ be defined on $\mathcal{P}(X)$ by

$$m(E) = \begin{cases} \text{card}(E), & E \subset A, \\ \text{card}(E) + 1, & E \subset B, E \neq \varnothing \\ -n, & \text{else}. \end{cases}$$

The set function $m$ is absolutely monotone, but it is not strictly absolutely monotone.

A signed fuzzy measure is not an absolutely monotone set function in general, as it is illustrated in the next example from [13].

**Example 2**: Let $X = \{1,2\}$ and let $m$ be a signed fuzzy measure defined on $\mathcal{P}(X)$ by: $m(\varnothing) = 0$, $m(\{1\}) = 3$, $m(\{2\}) = -2$, $m(\{1,2\}) = 1$. We have $m(\{1,2\}) < m(\{1\})$, hence $m$ is not absolutely monotone.

For a set function $m$ we define its positive and negative part $m^+, m^- : \mathcal{A} \to [0,a]$ by:

$$m^+(A) = m(A) \vee 0, \quad m^-(A) = (-m(A)) \vee 0.$$ 

Maxitive fuzzy measures are considered in [16], [22].

**Definition 4**: A fuzzy measure $m$ is completely maxitive if for any family $\{A_i\}_{i \in I}$ from $\mathcal{A}$ such that $\bigcup_{i \in I} A_i \in \mathcal{A}$ we have

$$m\left( \bigcup_{i \in I} A_i \right) = \sup_{i \in I} m(A_i).$$

$\oplus$-measures are introduced in [11].

**Definition 5**: A set function, $m : \mathcal{A} \to [-a,a]$, is a $\oplus$-measure if it satisfies

$$m(A \cup B) = m(A) \oplus m(B)$$

for all $A, B \in \mathcal{A}$, $A \cap B = \varnothing$.

Note that each $\oplus$-measure is revised monotone (Definition 2(ii)), hence it is a signed fuzzy measure.

**Example 3**: Let $X = \{x_1, x_2, \ldots, x_n\}$. Let $m$ be a set function $m : \mathcal{P}(X) \to [-1,1]$ with $m(\varnothing) = 0$, defined by:

$$m(A) = \begin{cases} \frac{1}{\min_{x_i \in A} i} & \text{if } \min_{x_i \in A} i = 2k \\ -\frac{1}{\min_{x_i \in A} i} & \text{if } \min_{x_i \in A} i = 2k + 1 \end{cases}$$

$m$ is a $\oplus$-measure.

### III. Space of real set functions $AMSS$

In this section we present results proven in [13] concerning to the space $AMSS$ and the representation of a real-valued set function from $AMSS$ by a symmetric maximum of two monotone set functions.

**Definition 6**: Let $m : \mathcal{A} \to [-a,a]$, $a \in \mathbb{R}^+$, $m(\varnothing) = 0$ be a set function.

(i) We say $m$ is a sign stable if it fulfils:

$$\sup_{E \subset A} m^+(E) < m^-(A), \quad \text{if } m(A) < 0,$$

$$\sup_{E \subset A} m^+(E) < m^+(A), \quad \text{if } m(A) > 0,$$

for all $E \subset A$.

(ii) If $m$ is sign stable and absolutely monotone, we say $m$ belongs to the class $AMSS$.

**Lemma 1**: Let $m_1, m_2 : \mathcal{A} \to [0,a]$ be fuzzy measures such that for each $A \subset B$, $A, B \in \mathcal{A}$ we have $m_1(A) = m_2(A)$, whenever $m_1(B) = m_2(B)$. Then a set function $m : \mathcal{A} \to [-a,a]$ defined by $m(A) := m_1(A) \ominus (m_2(A))$, for $A \in \mathcal{A}$ is absolutely monotone.

**Proposition 1**: Each absolutely strictly monotone set function defined on a finite $\mathcal{A}$, belongs to $AMSS$.

We have the next representation theorem of a set function from $AMSS$.

**Theorem 1**: Let $m : \mathcal{A} \to [-a,a]$, $a \in \mathbb{R}^+$ be from $AMSS$, such that $m(\varnothing) = 0$. Then, there exist two fuzzy measures $m_1, m_2 : \mathcal{A} \to [0,a]$ such that $m = m_1 \ominus (m_2)$ and $|m| = m_1 \vee m_2$, where $m_1 : \mathcal{A} \to [0,a]$ is defined by:

$$m_1(A) = \begin{cases} m^+(A), & m(A) \geq 0, \\ \sup_{E \subset A} m^+(E), & m(A) < 0 \end{cases}$$
and $m_2 : A \rightarrow [0, a]$ is defined by:

$$m_2(A) = \begin{cases} m^-(A), & m(A) \leq 0, \\ \sup_{E \in A} m^-(E), & m(A) > 0. \end{cases}$$  \hspace{1cm} (7)

**Example 4:** Let $m$ be a set function defined on the subclass of Borel subsets of the interval $[-a, a]$. For each $[c, d] \in \mathcal{B}([-a, a])$, $-a \leq c \leq d \leq a, a \in \mathbb{R}^+$ $m$ is defined by:

$$m([c, d]) = \begin{cases} d - c, & d > 0, \\ c - d, & d \leq 0. \end{cases}$$

$m$ is an absolutely monotone set function. If $a < \infty$, then $m$ is a sign stable on class of Borel subsets having form $[c, d]$. Hence, there exist $m_1$ and $m_2$ defined with (6), (7), given by:

$$m_1([c, d]) = \begin{cases} d - c, & d > 0, \\ 0, & d \leq 0, \end{cases}$$

$$m_2([c, d]) = \begin{cases} 0, & d > 0, c > 0 \\ -c, & d > 0, c \leq 0 \\ -d, & d \leq 0. \end{cases}$$

Obviously, $m_1$ and $m_2$ are fuzzy measures and they induce measures $m_1$ and $m_2$ on $\mathcal{B}([-a, a])$. Together with Theorem 1 this fact implies

$$m(B) = m_1(B) \otimes (-m_2(B)), \quad \text{for each } B \in \mathcal{B}([-a, a]).$$

**Theorem 2:** Let $m : A \rightarrow [-a, a], a \in \mathbb{R}^+$, such that $m(\emptyset) = 0$.

(i) $m \in AMSS$ if and only if there exist fuzzy measures $m_1, m_2 : A \rightarrow [0, a]$, such that for each $A \subset B, A, B \in A$ we have $m_1(A) = m_2(A)$ whenever $m_1(B) = m_2(B)$, and $m = m_1 \otimes (-m_2)$.

(ii) Moreover, if $m \in AMSS$ and $m = m_1 \otimes (-m_2)$ for some fuzzy measures $m_1, m_2$, then $m_1 \leq m_1$ and $m_2 \leq m_2$, where $m_1$ and $m_2$ are defined by (6) and (7).

**Example 5:** Let $X = \{1, 2, 3, 4\}$ and $m$ be a $\otimes$-measure such that $m(\emptyset) = 0$, defined on $\mathcal{P}(X)$ by:

| $m(\{1\})$ | $m(\{2\})$ | $m(\{3\})$ | $m(\{4\})$ | $m(\{1,2\})$ | $m(\{1,3\})$ | $m(\{1,4\})$ | $m(\{2,3\})$ | $m(\{2,4\})$ | $m(\{1,2,3\})$ | $m(\{1,2,4\})$ | $m(\{1,3,4\})$ | $m(\{2,3,4\})$ | $m(\{3,4\})$ | $m(\{1,3,4\})$ |
|----------|----------|-----------|----------|------------|-------------|-------------|------------|-------------|----------------|----------------|--------------|----------------|-------------|-----------|----------|
| 0        | -3       | -4       | 5        | -3         |             | 5           | -4         | 5           | -4            | 5              |             | -4            | 5           |          |          |

$m \in AMSS$ and the lowest fuzzy measures (in the sense of Theorem 2 (ii)) such that $m = m_1 \otimes (-m_2)$ are $m_1$ and $m_2$ given by (6) and (7):

$$m_1(A) = m^+(A) \quad \text{for all } A \in \mathcal{P}(X) \quad \text{and}$$

The relationship of $m \in AMSS$ and a $\otimes$-measure is established in the next theorems. In the sequel $A = \mathcal{P}(X)$.

**Theorem 3:** Let $m \in AMSS, m(\emptyset) = 0$. If $m_1$ and $m_2$ given by Theorem 1 are completely maxitive fuzzy measures defined by (6) and (7), respectively, then $m$ is a $\otimes$-measure.

**Theorem 4:** Let $m$ be a $\otimes$-measure such that each subset of $m$-null-set is $m$-null-set, i.e. for each $A \subset B, A, B \in A$ we have $m(A) = 0$, whenever $m(B) = 0$. Then $m \in AMSS$.

**IV. INTEGRALS BASED ON $m \in AMSS$**

Let $(X, A)$ be a measurable space, where $X$ is a universal set. Let $f : X \rightarrow [-1, 1]$ be an $A$-measurable, such that $\sup f(x) < 1$. We denote with $\mathcal{F}$ the class of such functions.

We propose three different types of integrals of function $f \in \mathcal{F}$. The first one is related to the Sugeno integral, the second is related to the Choquet-like integral, and the third one can be obtained as a limit of the sequence of integrals of second type.

Let $g : [-1, 1] \rightarrow [-\infty, \infty], g(0) = 0$ be an odd, strictly increasing, continuous function. A pseudo-addition $\oplus : [-1, 1]^2 \rightarrow [-1, 1]$ is defined by

$$x \oplus y = g^{-1}(g(x) + g(y));$$  \hspace{1cm} (8)

with the convention $-\infty - \infty = \infty$ or $\infty - \infty = -\infty$.

A pseudo-addition $\oplus$ is associative, commutative, strictly monotone on $[-1, 1]^2$, continuous up to $(1, 1), (-1, -1)$, with neutral element 0. It is a symmetric operation, i.e. for all $x, y \in [-1, 1]^2 \setminus \{(1, 1), (1, -1)\}$ we have

$$-(x \oplus y) = (-x) \oplus (-y).$$

The pseudo-difference $\ominus$ with respect to $\oplus$ for $x, y \in [-1, 1]^2 \setminus \{(1, 1), (1, -1)\}$ is given by

$$x \ominus y = x \oplus (-y).$$

A pseudo-multiplication $\odot : [-1, 1]^2 \rightarrow [-1, 1]$, distributive with respect to $\oplus$, can be defined by the additive generator of pseudo-addition $\oplus$, see [6], [7], [10], [12]. The pseudo-multiplication $\odot : [-1, 1]^2 \rightarrow [-1, 1]$ is defined by:

$$x \odot y = g^{-1}(g(x)g(y)),$$  \hspace{1cm} (9)

with the convention $\infty \cdot 0 = 0$ or $0 \cdot \infty = \infty$.

**Definition 7:** Let $m \in AMSS, |m(X)| < 1$ and $f \in \mathcal{F}$. We define

(1) Asymmetric Sugeno integral of $f$ based on $m$:

$$S_m(f) = S_m(f^+) \ominus (S_m(f^-)),$$
(2) Generated Choquet integral of $f$ based on $m$: 
\[ C^g_m(f) = C^g_{m_1}(f^+) \ominus C^g_{m_2}(f^-), \]
(3) Asymmetric $(\ominus, \circ)$-integral of $f$ based on $m$:
\[ ASI_m(f) = \sup_{t \in [0,1]} \left( t \odot m_1(\{f^+ \geq t\}) \ominus \odot \left( -\sup_{t \in [0,1]} (t \odot m_2(\{f^- \geq t\})) \right) \right), \]
where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$.

**Proposition 2:** Let $m \in AMSS$, such that $|m(X)| < 1$ and let $g : [-1, 1] \to [-\infty, \infty]$, $g(0) = 0$ be an odd, strictly increasing, continuous function. Then there exists a sequence of functions $\{g_n\}_{n \in \mathbb{N}}$ such that for all $f \in \mathcal{F}$ we have
\[ \lim_{n \to \infty} C^g_n(f) = ASI_m(f). \]

**Proposition 3:** The integrals proposed in Definition 7 are monotone and for all $m \in AMSS$, $|m(X)| < 1$ and $f \in \mathcal{F}$ we have
\[ I_m(-f) = -I_m(f), \]
where $-m \in AMSS$ and $|-m(X)| < 1$.

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**References**