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IMAGE SEGMENTATION BY APPLYING MEDIAN-AGGREGATED DISTANCE FUNCTIONS

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Abstract. In this paper we present construction of new distance functions by using aggregation operators of *median* type on given sequence of distance functions. Depending on characteristics of the given distance functions, features of new constructed function are analyzed. Also, one application on image segmentation is presented.

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1. Introduction

Distance functions and metrics have application in various scientific disciplines, and one of them is automatic image segmentation. In this paper, we consider construction of new distance function by applying aggregation operator *median* Med on some given distance functions, and one application of such function on image segmentation by using *Fuzzy c-means clustering algorithm* (shortly FCM), see [1]. In paper [2], it is shown that by applying aggregation operator on distance functions, new distance function is constructed. With certain additional properties of applied aggregation operator, constructed distance function could inherit certain good, and for application useful properties from initial given functions. In [2], from this aspect, one class of aggregation operators was considered, so called *generalized means*.

Aggregation operators are type of fuzzy operations, which have found large application in various engineering disciplines, see [3].

Definition 1.1. *Aggregation operator* is function $A : \bigcup_{n=2}^{\infty} [0,1]^n \to [0,1]$ with following properties.

(a01) For each $n \geq 2$ and n-tuples $(0,\dots,0) \in [0,1]^n$, $(1,\dots,1) \in [0,1]^n$, *boundary conditions* $A(0,\dots,0) = 0$ and $A(1,\dots,1) = 1$ are satisfied.

(a02) Function A is monotonically non-decreasing on each component, i.e. for each $n \geq 2$ and n-tuples $(a_1,\dots,a_n) \in [0,1]^n$, $(b_1,\dots,b_n) \in [0,1]^n$

$$\forall i \in \{1,\dots,n\},\ a_i \leq b_i \ \Rightarrow\ A(a_1,\dots,a_n) \leq A(b_1,\dots,b_n)$$

holds.

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In addition, function A could have some of the following additional properties.

(a03) Function A is continuous.

(a04) Function A is symmetric, i.e. for each $n \geq 2$, for each $(a_1, \ldots, a_n) \in [0,1]^n$ and each permutation p of set $\{1, \ldots, n\}$ applies

$$A(a_1, \ldots, a_n) = A(a_{p(1)}, \ldots, a_{p(n)}).$$

(a05) Function A is idempotent, i.e. for $n \geq 2$ and $(a, \ldots, a) \in [0,1]^n$ applies

$$A(a, \ldots, a) = a.$$

(a06) Function A is additive, i.e. for each $n \geq 2$ and each $(a_1, \ldots, a_n) \in [0,1]^n$, $(b_1, \ldots, b_n) \in [0,1]^n$ with a property $(a_1+b_1, \ldots, a_n+b_n) \in [0,1]^n$ applies

$$A(a_1 + b_1, \ldots, a_n + b_n) = A(a_1, \ldots, a_n) + A(b_1, \ldots, b_n).$$

(a07) Function A is sub-additive, i.e. for $n \geq 2$ and each $(a_1, \ldots, a_n) \in [0,1]^n$, $(b_1, \ldots, b_n) \in [0,1]^n$ with a property $(a_1+b_1, \ldots, a_n+b_n) \in [0,1]^n$ applies

$$A(a_1 + b_1, \ldots, a_n + b_n) \leq A(a_1, \ldots, a_n) + A(b_1, \ldots, b_n).$$

(a08) Function A is positively homogeneous, i.e. for each $t \geq 0$, each $n \geq 2$ and all $(a_1, \ldots, a_n) \in [0,1]^n$ with a property $(ta_1, \ldots, ta_n) \in [0,1]^n$ applies

$$A(ta_1, \ldots, ta_n) = tA(a_1, \ldots, a_n).$$

(a09) Function A is positively sub-homogeneous, i.e. for each $t \geq 0$, $n \geq 2$ and all $(a_1, \ldots, a_n) \in [0,1]^n$ with a property $(ta_1, \ldots, ta_n) \in [0,1]^n$ applies

$$A(ta_1, \ldots, ta_n) \leq tA(a_1, \ldots, a_n).$$

When constructing new distance functions by applying aggregation operators to ordinary distance functions, the following properties that the aggregation operator $A : \bigcup\limits_{n=2}^{\infty} [0,1]^n \to [0,1]$ could have are also of interest.

(a10) For each $n \geq 2$ applies: $\quad A(a_1, \ldots, a_n) = 0 \;\Rightarrow\; \forall i \in \{1, \ldots, n\},\ a_i = 0.$

(a11) For each $n \geq 2$ applies: $\quad A(a_1, \ldots, a_n) = 0 \;\Rightarrow\; \exists i \in \{1, \ldots, n\},\ a_i = 0.$

Distance functions could be interpreted as a measure of difference between two objects.

Definition 1.2. Let $X \neq \emptyset$ be an arbitrary set. A *distance function* on set (space) X is a function $d : X^2 \to [0, \infty)$, which have the following properties.

(d01) $\forall x, y \in X,\ \ d(x,y) = d(y,x),$ (*symmetry*)

(d02) $\forall x \in X,\ \ d(x,x) = 0.$ (*reflexivity*)

Ordered pair (X, d) is then called *a space with a distance*.

Definition 1.3. In space $X \neq \emptyset$, function $d : X^2 \to [0, \infty)$ could have following important properties.

(d03) $\forall x, y \in X,\ \ d(x,y) = 0 \;\Rightarrow\; x = y$ (*identity of indiscernibles*).

(d04) $\forall x, y, z \in X,\ \ d(x,z) \leq d(x,y) + d(y,z)$ (*triangle inequality*).

(d05) For certain constant $C \in [1, \infty)$ applies:

$\forall x, y, z \in X,\ \ d(x,z) \leq C\,(d(x,y) + d(y,z))$ (*C-triangle inequality*).

(d06) $d : X^2 \to [0,1] \ \ \vee \ \ \exists a > 0,\ d : X^2 \to [0,a]$ (*boundedness*).

A distance function $d : X^2 \to [0, \infty)$ is a *metric* on a set X if satisfies identity of indiscernibles (d03) and triangle inequality (d04).

2. Previous research

Let $d_i : X^2 \to [0,1]$, $i \in \mathbb{N}$ be bounded distance functions on set $X \neq \emptyset$, and $A : \bigcup_{n=2}^{\infty} [0,1]^n \to [0,1]$ be an arbitrary aggregation operator. Let function $d : X^2 \times \mathbb{N} \setminus \{1\} \to [0,1]$ be defined with

$$d(x,y;n) = A\left(d_1(x,y), \dots, d_n(x,y)\right), \quad x,y \in X, \quad n \in \mathbb{N} \setminus \{1\},$$

and for each $n \geq 2$ function $d_{[n]} : X^2 \to [0,1]$ be defined with

$$d_{[n]}(x,y) = d(x,y;n) = A\left(d_1(x,y), \dots, d_n(x,y)\right), \quad x,y \in X.$$

In the following theorem (see [2]), certain properties of function $d_{[n]}$ are stated, depending on properties of distance function d_i and properties of aggregation operator A, primarily (a07), (a09), (a10) and (a11) from section 1.

Theorem 2.1. *Let $d_i : X^2 \to [0,1]$, $i \in \mathbb{N}$ be arbitrary sequence of distance functions, and let function $A : \bigcup_{n=2}^{\infty} [0,1]^n \to [0,1]$ be an arbitrary aggregation operator. Then for each $n \geq 2$, and for all functions $d_{[n]} : X^2 \to [0,1]$ defined with $d_{[n]}(x,y) = A\left(d_1(x,y), \dots, d_n(x,y)\right)$, $x,y \in X$ following statements holds.*

(ad01) *Function $d_{[n]}$ is distance function.*

(ad02) *If for each of distance function d_i, $i \in \mathbb{N}$ applies identity of indiscernibles* (d03) *and operator A has property* (a11) *for each $n \geq 2$, then for function $d_{[n]}$ applies identity of indiscernibles* (d03).

(ad03) *If for at least one distance function d_i, $i \in \{1, \dots, n\}$ applies identity of indiscernibles* (d03) *and operator A has property* (a10) *for each $n \geq 2$, then for function $d_{[n]}$ applies identity of indiscernibles* (d03).

(ad04) *Let all d_i, $i \in \mathbb{N}$ be metric, and let $A : \bigcup_{n=2}^{\infty} [0,\infty)^n \to [0,\infty)$ be sub-additive function (property* (a07)*), which restriction on set $\bigcup_{n=2}^{\infty} [0,1]^n$ is an aggregation operator. If the aggregation operator $A : \bigcup_{n=2}^{\infty} [0,1]^n \to [0,1]$ has property* (a11) *for each $n \geq 2$, then function $d_{[n]}$ is metric for each $n \geq 2$.*

(ad05) *Let for each function d_i, $i \in \mathbb{N}$ applies C-triangle inequality* (d05). *Let $A : \bigcup_{n=2}^{\infty} [0,\infty)^n \to [0,\infty)$ be function which restriction on the set $\bigcup_{n=2}^{\infty} [0,1]^n$ is aggregation operator that is sub-additive and positively sub-homogeneous, i.e. has properties* (a07) *and* (a09). *Then for functions $d_{[n]}$, $n \geq 2$ applies C-triangle inequality* (d05). *Therefore, if for each function d_i, $i \in \mathbb{N}$ C-triangle inequality applies with corresponding constant C_i, $i \in \mathbb{N}$, then C-triangle inequality for functions $d_{[n]}$, $n \geq 2$ applies with corresponding constants $C_{[n]} = \max\{C_1, \dots, C_n\}$, $n \geq 2$.*

(ad06) *Let A be continuous aggregation operator. If space X is equipped with topological structure and each of distance function d_i, $i \in \mathbb{N}$ is continuous on X^2, then distance function $d_{[n]}$ is continuous on X^2 for $n \geq 2$.*

3. Certain properties of median operator

Let us test properties (a03)...(a11) of aggregation operator *statistic median* $\mathsf{med} : \bigcup_{n=2}^{\infty} [0,1]^n \to [0,1]$ defined as following. For arbitrary n-tuple $(x_1, \ldots, x_n) \in [0,1]^n$ let us denote $(x'_1, \ldots, x'_n)$ as non-decreasing permutation of $(x_1, \ldots, x_n)$. Then

$$\mathsf{med}\,(x_1, \ldots, x_n) = \begin{cases} \dfrac{x'_{\frac{n}{2}} + x'_{\frac{n}{2}+1}}{2} & , \quad \exists k \in \mathbb{Z},\ n = 2k \\ x'_{\frac{n+1}{2}} & , \quad \exists k \in \mathbb{Z},\ n = 2k+1 \end{cases}.$$

It is known that med is continuous, symmetric an idempotent operator. It easily can be verified that med is positively homogeneous, and therefore sub-homogeneous (properties (a09) and (a08)). It does not have a property (a10) because e.g. $\mathsf{med}\,(0,0,1) = 0$, but it does have a property (a11) because arguments of operator med are from $[0,1]$. Following example shows that med is not sub-additive (property (a07)), nor additive (property (a06)). E.g. for $(a_1, a_2, a_3) = (0.2, 0.4, 0.2)$ and $(b_1, b_2, b_3) = (0.2, 0.5, 0.6)$ holds

$$\begin{aligned} \mathsf{med}\,(a_1 + b_1, a_2 + b_2, a_3 + b_3) &= \mathsf{med}\,(0.4, 0.9, 0.8) = 0.8, \\ \mathsf{med}\,(a_1, a_2, a_3) + \mathsf{med}\,(b_1, b_2, b_3) &= \mathsf{med}\,(0.2, 0.4, 0.2) + \mathsf{med}\,(0.2, 0.5, 0.6) \\ &= 0.2 + 0.5 = 0.7 \\ &< \mathsf{med}\,(a_1 + b_1, a_2 + b_2, a_3 + b_3). \end{aligned}$$

In summary, considered properties are presented in the following table.

	(a03)	(a04)	(a05)	(a06)	(a07)	(a08)	(a09)	(a10)	(a11)
med	YES	YES	YES	no	no	YES	YES	no	YES

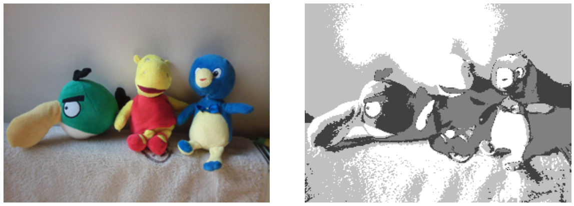
Also, considering properties of operator med, following table shows in which statements of the theorem 2.1 this operator could be applied.

	(ad01)	(ad02)	(ad03)	(ad04)	(ad05)	(ad06)
med	YES	YES	no	no	no	YES

4. Application of median operator in image segmentation

This section shows application of median-aggregated distance function in image segmentation by using *Fuzzy c-Means Clustering Algorithm* (FCM), see [1]. Input parameters of algorithm, besides an image, are listed bellow.

- Selected number $c = 4$, present number of clusters that are going to be acquired by segmentation.
- Weight coefficient $m = 2.0$, present parameter which choice could affect on quality and speed of segmentation.
- Distance function $d : P \times P \to [0, \infty)$, where P is a set of pixels from selected image, defines a segmentation criterion. For pixels $p_1, p_2 \in P$,



(a) Original image. (b) d_{Med} segmentation.

Figure 1: Original picture and median-segmentation.

value $d(p_1, p_2) \in [0, \infty)$ represents a measure of difference between them, and with appropriate selection of function d a criterion of pixel grouping into clusters is chosen. In order to compare achieved results for used distance function, all distance functions are normalized i.e. modified to $d : P \times P \to [0, 1]$. In paper [2] the following statement is proved. Let $\tilde{d} : P \times P \to [0, a]$ be bounded function for some constant $a > 0$, and let function $d : P \times P \to [0, \infty)$ be defined with $d(p_1, p_2) = \dfrac{1}{a}\tilde{d}(p_1, p_2)$ for $p_1, p_2 \in P$. Then $d : P \times P \to [0, 1]$ is normalized function, which inherits all properties (d01)...(d06) from function $\tilde{d}$. Accordingly, if function $\tilde{d} : P \times P \to [0, a]$ is distance function, then $d : P \times P \to [0, 1]$ is distance function. Also, if $\tilde{d} : P \times P \to [0, a]$ is metrics, then $d : P \times P \to [0, 1]$ is metrics, etc.

Besides the segmented image obtained, as the output parameters of the algorithm, the following values are presented.

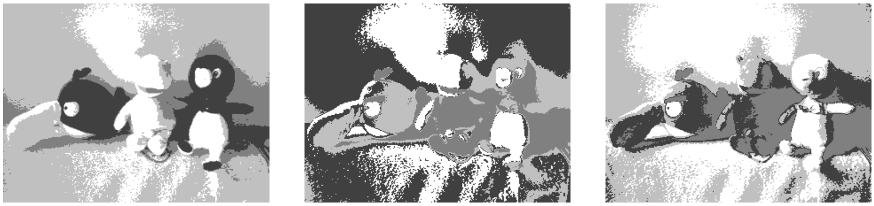
- PI - *performance index* which measures the clustering of data. The lower value of the PI coefficient represents stronger grouping of pixels, i.e. more compact clusters.
- IT - number of iterations.
- RT - algorithm operating time, in seconds.

Applied FCM algorithm is coded in MATLAB, R2012b, 32-bit (win32). It is tested on PC with Intel(R) processor Core(TM)2 Duo CPU E8400 3.00 GHz, 3.25 GB of RAM, and operating system Microsoft Windows XP, Professional, Version 2002.

Application of distance function constructed with med aggregation operator is presented on segmentation of color image, Figure 1a. Image is in BMP format, 225×300 pixels in size, with 3-byte RGB color coding. Let $p = (r, g, b) \in P$ be the pixel where $r, g, b \in \{0, \dots, 255\}$ are codes for red, green and blue color component of pixel, respectively. Therefore, $P = \{0, \dots, 255\}^3$. Image presented on Figure 1a is initially segmented by using normalized distance functions

$$d_R(p_1, p_2) = \tfrac{1}{255} \cdot |r_2 - r_1|, \quad p_1 = (r_1, g_1, b_1) \in P,\ p_2 = (r_2, g_2, b_2) \in P,$$
$$d_G(p_1, p_2) = \tfrac{1}{255} \cdot |g_2 - g_1|, \quad p_1 = (r_1, g_1, b_1) \in P,\ p_2 = (r_2, g_2, b_2) \in P,$$
$$d_B(p_1, p_2) = \tfrac{1}{255} \cdot |b_2 - b_1|, \quad p_1 = (r_1, g_1, b_1) \in P,\ p_2 = (r_2, g_2, b_2) \in P,$$

which can be treated as L_1 metrics on set $\{0, \dots, 255\}$ of corresponding color component. These metrics define segmentation criteria on single color component. Application of these distances in FCM algorithm gives following images



(a) d_R segmentation. (b) d_G segmentation. (c) d_B segmentation.

Figure 2: Segmentations by using initial distance functions.

2a, 2b i 2c. Then, the image is segmented by using distance function that is constructed by applying aggregation operator med to d_R, d_G and d_B:

$$d_{\mathsf{med}}\left(p_1,p_2\right)=\mathsf{med}\left(d_R\left(p_1,p_2\right),d_G\left(p_1,p_2\right),d_B\left(p_1,p_2\right)\right),\quad p_1,p_2\in P,$$

which can be interpreted as median value of difference between color components. Application of med operator on distance functions d_R, d_G and d_B gives new distance function d_{med} (see [2]). Segmentation results for original image, Figure 1a, by using d_{med} distance function is shown on Figure 1b. Table 1 shows output parameters of applied segmentations with different distance functions d_R, d_G, d_B i d_{med}. Application of d_{med} distance function does not give better

	d_R	d_G	d_B	d_{med}
PI:	147.4538	141.8972	105.4255	1.2378e+07
IT:	27	21	25	27
RT:	88	73	79	204

Table 1: Output parameters of applied segmentations.

parameters PI, IT and RT in comparison to values obtained with d_R, d_G and d_B. However, on the author's opinion, visual comparison of images 1b, 2a, 2b and 2c shows that on image 1b regions are "better" segmented, in a way that are more appealing to human eye.

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