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WEIGHTED MINIMUM AND WEIGHTED MAXIMUM

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Abstract. In this paper we present weighted minimum and weighted maximum functions. We prove that they are aggregation functions, which special cases are ordinary minimum and maximum aggregation functions, and that weighted minimum and weighted maximum are continuous and idempotent, but not symmetrical aggregation functions. Additionally, we present some properties of ordinary minimum and maximum aggregation functions which are relevant for construction and properties of constructed distance functions, metrics, fuzzy-measures and other measure-type functions.

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1. Introduction

This paper is continuations of research carried out [6, 7, 8]. Minimum and maximum are basic fuzzy operations, minimum as fuzzy conjunction or fuzzy set intersection, and maximum as fuzzy disjunction or fuzzy set union. In research papers [6, 7, 8], some properties of minimum and maximum were investigated, as their applicability in construction of distance functions which are further used in image segmentation. Their properties can be relevant also in construction of other measure-type functions. In the Section 3 we present a generalization of minimum and maximum aggregation functions, so called weighted minimum and weighted maximum functions. In the Section 3 we prove that weighted minimum and weighted maximum are aggregation functions which special cases are ordinary minimum and maximum, and that weighted minimum and weighted maximum are continuous and idempotent, but not symmetrical aggregation functions.

2. Preliminaries and previous research

In this section we list the definition of aggregation function, see [1, 2, 5], some its additional properties which are relevant for aggregation of distance functions and measure-type functions, see [3, 8]. We also present which of mentioned properties satisfies minimum and maximum aggregation function.

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Definition 2.1 (Aggregation function)**.** For fixed $n \in \mathbb{N}$, an n-ary *aggregation function* is a function $A_{[n]} : [0,1]^n \to [0,1]$ with the following properties.

(a01) Boundary conditions $A_{[n]}(0,\dots,0) = 0$ and $A_{[n]}(1,\dots,1) = 1$ hold.

(a02) Function $A_{[n]}$ is monotonically non-decreasing in each component, i.e.,
$$\forall i \in \{1,\dots,n\}\,,\ a_i \le b_i \quad \Rightarrow \quad A_{[n]}(a_1,\dots,a_n) \le A_{[n]}(b_1,\dots,b_n)$$
hold for arbitrary n-tuples $(a_1,\dots,a_n) \in [0,1]^n$, $(b_1,\dots,b_n) \in [0,1]^n$.

In case $n = 1$, $A_{[1]}(x) = x$ for all $x \in [0,1]$.

Definition 2.2 (Extended aggregation function)**.** An *extended aggregation function* is a mapping $A : \bigcup\limits_{n=1}^{\infty} [0,1]^n \to [0,1]$ such that the every restriction $A_{[n]} : [0,1]^n \to [0,1]$, $n \in \mathbb{N}$ of this mapping is an n-ary aggregation function.

The following properties of aggregation functions can be relevant for properties of distance functions and fuzzy measures constructed by applying aggregation function on sequence of initial functions of same type.

Definition 2.3. Let $n \in \mathbb{N}$, and let $A_{[n]} : [0,1]^n \to [0,1]$ be an n-ary aggregation function. It may have some of following properties.

(a03) $A_{[n]}$ is continuous.

(a04) $A_{[n]}$ is symmetric in each component, i.e., for each $(a_1,\dots,a_n) \in [0,1]^n$ and each permutation p of the set $\{1,\dots,n\}$ hold
$$A_{[n]}(a_1,\dots,a_n) = A_{[n]}(a_{p(1)},\dots,a_{p(n)}).$$

(a05) Aggregation function $A_{[n]}$ is idempotent, i.e., for all $(a,\dots,a) \in [0,1]^n$ hold
$$A_{[n]}(a,\dots,a) = a.$$

(a06) $A_{[n]}$ is additive, i.e., for all $(a_1,\dots,a_n), (b_1,\dots,b_n) \in [0,1]^n$ that satisfy $(a_1+b_1,\dots,a_n+b_n) \in [0,1]^n$ hold
$$A_{[n]}(a_1+b_1,\dots,a_n+b_n) = A_{[n]}(a_1,\dots,a_n) + A_{[n]}(b_1,\dots,b_n).$$

(a07) $A_{[n]}$ is subadditive, i.e., for all $(a_1,\dots,a_n), (b_1,\dots,b_n) \in [0,1]^n$ that satisfy $(a_1+b_1,\dots,a_n+b_n) \in [0,1]^n$ hold
$$A_{[n]}(a_1+b_1,\dots,a_n+b_n) \le A_{[n]}(a_1,\dots,a_n) + A_{[n]}(b_1,\dots,b_n).$$

(a08) $A_{[n]}$ is superadditive, i.e., for all $(a_1,\dots,a_n), (b_1,\dots,b_n) \in [0,1]^n$ that satisfy $(a_1+b_1,\dots,a_n+b_n) \in [0,1]^n$ hold
$$A_{[n]}(a_1+b_1,\dots,a_n+b_n) \ge A_{[n]}(a_1,\dots,a_n) + A_{[n]}(b_1,\dots,b_n).$$

(a09) Aggregation function $A_{[n]}$ is positively homogeneous, i.e., for each $t \ge 0$ and all $(a_1,\dots,a_n) \in [0,1]^n$ that satisfy $(ta_1,\dots,ta_n) \in [0,1]^n$ hold
$$A_{[n]}(ta_1,\dots,ta_n) = tA_{[n]}(a_1,\dots,a_n).$$

(a10) Aggregation function $A_{[n]}$ is positively subhomogeneous, i.e., for each $t \geq 0$ and all $(a_1, \ldots, a_n) \in [0,1]^n$ that satisfy $(ta_1, \ldots, ta_n) \in [0,1]^n$ hold

$$A_{[n]}(ta_1, \ldots, ta_n) \leq tA_{[n]}(a_1, \ldots, a_n).$$

(a11) Aggregation function $A_{[n]}$ is positively superhomogeneous, i.e., for each $t \geq 0$ and all $(a_1, \ldots, a_n) \in [0,1]^n$ that satisfy $(ta_1, \ldots, ta_n) \in [0,1]^n$ hold

$$A_{[n]}(ta_1, \ldots, ta_n) \geq tA_{[n]}(a_1, \ldots, a_n).$$

(a12) $A_{[n]}(a_1, \ldots, a_n) = 0 \;\Rightarrow\; \forall i \in \{1, \ldots, n\}\,,\; a_i = 0.$

(a13) $A_{[n]}(a_1, \ldots, a_n) = 0 \;\Rightarrow\; \exists i \in \{1, \ldots, n\}\,,\; a_i = 0.$

(a14) $A_{[n]}(a_1, \ldots, a_n) < 1 \;\Rightarrow\; \forall i \in \{1, \ldots, n\}\,,\; a_i < 1.$

Ordinary minimum and maximum are well known aggregation functions and basic fuzzy operations. For fixed $n \geq 2$, functions

$\min_{[n]}(a_1, \ldots, a_n) = \min(a_1, \ldots, a_n), \quad (a_1, \ldots, a_n) \in [0,1]^n,$

$\max_{[n]}(a_1, \ldots, a_n) = \max(a_1, \ldots, a_n), \quad (a_1, \ldots, a_n) \in [0,1]^n,$

n-ary aggregation functions. Some of their properties from the Definition 2.3 are examined in [6, 7, 8], and examination of the rest of them is easy. Summary, considered properties of min and max function are shown in the Table 1.

	$\min_{[n]}$	$\max_{[n]}$
(a03)	YES	YES
(a04)	YES	YES
(a05)	YES	YES
(a06)	no	no
(a07)	no	YES
(a08)	YES	no
(a09)	YES	YES
(a10)	YES	YES
(a11)	YES	YES
(a12)	no	YES
(a13)	YES	YES
(a14)	no	YES

Table 1: Properties of $\min_{[n]}$ and $\max_{[n]}$.

3. Weighted minimum and maximum

In this section we analyze the properties from the Definition 2.3 of the weighted minimum and weighted maximum aggregation function, see [4]. Properties of ordinary minimum and maximum are analyzed in [7].

Definition 3.1. For $n \in \mathbb{N}$, let $\omega_{[n]} = (\omega_{n,1}, \ldots, \omega_{n,n}) \in [0,1]^n$ be an n-tuple of nonnegative coefficient from interval $[0,1]$ such that $\max\limits_{i \in \{1,\ldots,n\}} \omega_{n,i} = 1$, i.e.

$\omega_{n,i_0} = 1$ for some $i_0 \in \{1,\dots,n\}$. Function $\min_{[n]}^{\omega_{[n]}} : [0,1]^n \to [0,1]$ defined by

$$\min_{[n]}^{\omega_{[n]}}(a_1,\dots,a_n) = \min_{i\in\{1,\dots,n\}} \max(1-\omega_{n,i}, a_i) \tag{3.1}$$

for all $(a_1,\dots,a_n) \in [0,1]^n$. Function $\min_{[n]}^{\omega_{[n]}}$ is called *weighted minimum.*

Definition 3.2. For $n \in \mathbb{N}$, let $\omega_{[n]} = (\omega_{n,1},\dots,\omega_{n,n}) \in [0,1]^n$ be an n-tuple of nonnegative coefficient from interval $[0,1]$ such that $\max\limits_{i\in\{1,\dots,n\}} \omega_{n,i} = 1$, i.e. $\omega_{n,i_0} = 1$ for some $i_0 \in \{1,\dots,n\}$. Function $\max_{[n]}^{\omega_{[n]}} : [0,1]^n \to [0,1]$ defined by

$$\max_{[n]}^{\omega_{[n]}}(a_1,\dots,a_n) = \max_{i\in\{1,\dots,n\}} \min(\omega_{n,i}, a_i) \tag{3.2}$$

for all $(a_1,\dots,a_n) \in [0,1]^n$. Function $\max_{[n]}^{\omega_{[n]}}$ is called *weighted maximum.*

Remark 3.3. In the case when $\omega_{n,i} = 1$ for all $i \in \{1,\dots,n\}$, weighted minimum $\min_{[n]}^{\omega_{[n]}}$ reduces to ordinary min, and weighted maximum $\max_{[n]}^{\omega_{[n]}}$ reduces to ordinary max.

Theorem 3.4. *Functions* $\min_{[n]}^{\omega_{[n]}}$ *and* $\max_{[n]}^{\omega_{[n]}}$ *are aggregation functions for all* $n \in \mathbb{N}$.

Proof: First boundary condition $A_{[n]}(0,\dots,0) = 0$ from the Definition 2.1 is satisfied for $\min_{[n]}^{\omega_{[n]}}$ because

$$\min_{[n]}^{\omega_{[n]}}(0,\dots,0) = \min_{i\in\{1,\dots,n\}} \max(1-\omega_{n,i}, 0) = \min_{i\in\{1,\dots,n\}} (1-\omega_{n,i}) = 0$$

because $\omega_{n,i} = 1$ for at least one $i \in \{1,\dots,n\}$, i.e., $1-\omega_{n,i} = 0$ for some $i \in \{1,\dots,n\}$. It is also satisfied for $\max_{[n]}^{\omega_{[n]}}$ function because

$$\max_{[n]}^{\omega_{[n]}}(0,\dots,0) = \max_{i\in\{1,\dots,n\}} \min(\omega_{n,i}, 0) = \max_{i\in\{1,\dots,n\}} 0 = 0.$$

Second boundary condition $A_{[n]}(1,\dots,1) = 1$ from the definition 2.1 is satisfied for $\min_{[n]}^{\omega_{[n]}}$ because

$$\min_{[n]}^{\omega_{[n]}}(1,\dots,1) = \min_{i\in\{1,\dots,n\}} \max(1-\omega_{n,i}, 1) = \min_{i\in\{1,\dots,n\}} 1 = 1,$$

and also for $\max_{[n]}^{\omega_{[n]}}$,

$$\max_{[n]}^{\omega_{[n]}}(1,\dots,1) = \max_{i\in\{1,\dots,n\}} \min(\omega_{n,i}, 1) = 1$$

because $\omega_{n,i_0} = 1$ hold for some index $i_0 \in \{1,\dots,n\}$, and therefore is $\min(\omega_{n,i_0}, 1) = 1$.

Now we prove the monotonicity (a02) from the Definition 2.1 of function $\min_{[n]}^{\omega_{[n]}}$. Let $a_i \le b_i$ for all $i \in \{1,\dots,n\}$ and arbitrary $(a_1,\dots,a_n) \in [0,1]^n$ and $(b_1,\dots,b_n) \in [0,1]^n$. Then for all $i \in \{1,\dots,n\}$ hold

$$\max(1-\omega_{n,i}, a_i) \le \max(1-\omega_{n,i}, b_i),$$

and we obtain

$$\begin{aligned}\min_{[n]}^{\omega_{[n]}}(a_1,\dots,a_n) &= \min_{i\in\{1,\dots,n\}} \max(1-\omega_{n,i}, a_i)\\ &\le \min_{i\in\{1,\dots,n\}} \max(1-\omega_{n,i}, b_i)\end{aligned}$$

$$= \min{}_{[n]}^{\omega_{[n]}} (b_1, \ldots, b_n).$$

Monotonicity (a02) of function $\max{}_{[n]}^{\omega_{[n]}}$ follows analogously. For arbitrary $(a_1, \ldots, a_n) \in [0,1]^n$ and $(b_1, \ldots, b_n) \in [0,1]^n$, if $a_i \le b_i$ for all $i \in \{1, \ldots, n\}$ then for all $i \in \{1, \ldots, n\}$ hold

$$\min(\omega_{n,i}, a_i) \le \min(\omega_{n,i}, b_i),$$

and we obtain

$$\begin{aligned} \max{}_{[n]}^{\omega_{[n]}} (a_1, \ldots, a_n) &= \max_{i \in \{1,\ldots,n\}} \min(\omega_{n,i}, a_i) \\ &\le \max_{i \in \{1,\ldots,n\}} \min(\omega_{n,i}, b_i) \\ &= \max{}_{[n]}^{\omega_{[n]}} (b_1, \ldots, b_n). \end{aligned}$$

□

For weighted minimum and weighted maximum, let we consider properties (a03), (a04) and (a05) from the Definition 2.3. For $n = 1$, all properties are obviously satisfied.

Theorem 3.5. *Functions* $\min{}_{[n]}^{\omega_{[n]}}$ *and* $\max{}_{[n]}^{\omega_{[n]}}$ *are continuous, idempotent, but not symmetrical aggregation functions for all* $n \ge 2$.

Proof:

(a03) Functions $\min{}_{[n]}^{\omega_{[n]}}$ and $\max{}_{[n]}^{\omega_{[n]}}$ are continuous as composition of continuous operations.

(a04) Functions $\min{}_{[n]}^{\omega_{[n]}}$ and $\max{}_{[n]}^{\omega_{[n]}}$ are not symmetrical, except when $\omega_{n,i} = 1$ for all $i \in \{1, \ldots, n\}$, when they reduce to ordinary min and max aggregation functions. For example, for $n = 2$, $\omega_{[2]} = (0.5, 1)$ and $(a_1, a_2) = (0.8, 0.2)$ hold

$$\min{}_{[2]}^{\omega_{[2]}}(0.8, 0.2) = 0.2 \ne \min{}_{[2]}^{\omega_{[2]}}(0.2, 0.8) = 0.5,$$

$$\max{}_{[2]}^{\omega_{[2]}}(0.8, 0.2) = 0.5 \ne \max{}_{[2]}^{\omega_{[2]}}(0.2, 0.8) = 0.8.$$

(a05) We will prove that both functions $\min{}_{[n]}^{\omega_{[n]}}$ and $\max{}_{[n]}^{\omega_{[n]}}$ are idempotent.

$\min{}_{[n]}^{\omega_{[n]}}$: Let $a \in [0,1]$. From $\max_{i \in \{1,\ldots,n\}} \omega_{n,i} = 1$, i.e. $\omega_{n,i_0} = 1$ for some $i_0 \in \{1, ..., n\}$, follows $\max(1 - \omega_{n,i_0}, a) = a$ for $i_0 \in \{1, ..., n\}$, and $\min{}_{[n]}^{\omega_{[n]}}(a, \ldots, a) = \min_{i \in \{1,\ldots,n\}} \max(1 - \omega_{n,i}, a) \le a$. On the other side, for all $i \in \{1, ..., n\}$ hold $\max(1 - \omega_{n,i}, a) \ge a$, and therefore is $\min{}_{[n]}^{\omega_{[n]}}(a, \ldots, a) = \min_{i \in \{1,\ldots,n\}} \max(1 - \omega_{n,i}, a) \ge a$, so that $\min{}_{[n]}^{\omega_{[n]}}(a, \ldots, a) = a$ holds.

$\max{}_{[n]}^{\omega_{[n]}}$: Let $a \in [0,1]$. From $\max_{i \in \{1,\ldots,n\}} \omega_{n,i} = 1$, i.e. $\omega_{n,i_0} = 1$ for some $i_0 \in \{1, \ldots, n\}$, follows $\min(\omega_{n,i_0}, a) = a$ for $i_0 \in \{1, \ldots, n\}$, and therefore follows $\max{}_{[n]}^{\omega_{[n]}}(a, \ldots, a) = \max_{i \in \{1,\ldots,n\}} \min(\omega_{n,i}, a) \ge a$. On the other side, $\min(\omega_{n,i}, a) \le a$ for all $i \in \{1, \ldots, n\}$ and consequently $\max{}_{[n]}^{\omega_{[n]}}(a, \ldots, a) = \max_{i \in \{1,\ldots,n\}} \min(\omega_{n,i}, a) \le a$, so that $\min{}_{[n]}^{\omega_{[n]}}(a, \ldots, a) = a$ holds. □

4. Conclusions

We conclude that presented aggregation functions *weighted minimum* and *weighted maximum* are generalization of ordinary min and max aggregation functions. Interesting question for further research is which of considered properties of min and max hold also for *weighted minimum* and *weighted maximum*, and did they suitable for aggregation of distance functions, fuzzy measures, and other measure-type functions.

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