THE PSEUDO PROBABILITY

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Abstract

Let $(I, \oplus, \odot)$ be a semiring with a neutral element $\mathbf{0}$ of the $\oplus$ and a neutral element $\mathbf{1}$ of the $\odot$. Let Σ be a σ-algebra of subsets of nonempty set Ω. Pseudo probability is a function $\mathsf{P} : \Sigma \to I$ with the next properties: (1) $\mathsf{P}(\emptyset) = \mathbf{0}$ and $\mathsf{P}(\Omega) = \mathbf{1}$,
(2) $\mathsf{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} \mathsf{P}(A_i)$ for pairwise disjoint sets $\{A_i\}_{i\in\mathbb{N}}$ from Σ.

The triple $(\Omega, \Sigma, \mathsf{P})$ is called a pseudo probability space. In this paper will be present some properties of the pseudo probability space and the pseudo variables.

AMS Mathematics Subject Classification (2000): 28B15, 28E05, 60B10
Key words and phrases: $g-$ integral, pseudo-integral, pseudo - operation, semiring, quasi-arithmetic means, pseudo-expectation, pseudo-probability, pseudo - variable, convergence in the pseudo-probability, almost surely convergence, law of large numbers.

1 Preliminaries

We briefly present some notions from the pseudo-analysis (see [2]).
Let the order $\preceq$ be defined on a set $I \neq \emptyset$, and $\emptyset \neq I^* \subset I$.

Definition 1 *The* **pseudo-operation** *is a binary operation* $* : I \times I \to I$ *which is commutative, associative, positively nondecreasing (*$x \preceq y$ *implies* $x * u \preceq y * u$, $u \in I^*$*) and for which there exists a neutral element* e.

The element $u \in I$ is the **null element** of the operation $* : I^2 \to I$ if for any $x \in I,\ x * u = u * x = u$ holds.
Pseudo-operation $*$ is **idempotent** if for any $x \in I,\ x * x = x$ holds.

*Research supported by MNTRRS-1866.

Definition 2 *Let $\oplus$ and $\odot$ be two pseudo-operations defined on the ordered set $(I, \preceq)$, with $\mathbf{0}$ and $\mathbf{1}$ as neutral elements, respectively. Let $I^{\oplus} = I$, for the first operation, and $I^{\odot} = \{x \in I : \mathbf{0} \preceq x\}$, for the second operation. If $\odot$ is a distributive operation with respect to pseudo-operation $\oplus$ and $\mathbf{0}$ is a null element of the operation $\odot$, we say that the triplet $(I, \oplus, \odot)$ is a* **semiring**.

The semiring $(I, \oplus, \odot)$ will be denoted by $I^{\oplus,\odot}$.
Let I be a subinterval of $[-\infty, +\infty]$ (we will take usually closed subintervals). Then we name the operations $\oplus$ and $\odot$ **pseudo-addition** and **pseudo-multiplication**.
In this paper we will consider semirings with the following continuous operations:

Case I) a) (i) $\quad x \oplus y = \min(x, y) \qquad x \odot y = x + y$

on the interval $(-\infty, +\infty]$. We have $\mathbf{0} = +\infty$ and $\mathbf{1} = 0$. The idempotent operation min induces a partial (full) order in the following way: $x \leq y$ if and only if $\min(x, y) = y$. Hence this order is opposite to the usual order on the interval $(-\infty, +\infty]$. We denote this semiring by $(-\infty, +\infty]^{\min,+}$.

(ii) $\quad x \oplus y = \max(x, y) \qquad x \odot y = x + y$

on the interval $[-\infty, +\infty)$. We have $\mathbf{0} = -\infty$ and $\mathbf{1} = 0$.

b) (i) $\quad x \oplus y = \min(x, y) \qquad x \odot y = x \cdot y$

on the interval $(0, +\infty]$. We have $\mathbf{0} = \infty$ and $\mathbf{1} = 1$.

(ii) $\quad x \oplus y = \max(x, y) \qquad x \odot y = x \cdot y$

on the interval $[0, +\infty)$. We have $\mathbf{0} = 0$ and $\mathbf{1} = 1$.

Case II) Semirings with pseudo-operations defined by monotone and continuous generator g (see [4])

$$x \oplus y = g^{-1}(g(x) + g(y)) \qquad x \odot y = g^{-1}(g(x)g(y))$$

with the convention $0 \cdot (+\infty) = 0$, on the interval $[a, b]$. We have $\mathbf{0} = a$ or $\mathbf{0} = b$. This order is defined as $x \preceq y \Leftrightarrow g(x) \leq g(y)$.

Case III) a) Let $\oplus = \max$ and $\odot = \min$ on the interval $[-\infty, +\infty]$. We have $\mathbf{0} = -\infty$ and $\mathbf{1} = +\infty$.

b) Let $\oplus = \min$ and $\odot = \max$ on the interval $[-\infty, +\infty]$. We have $\mathbf{0} = +\infty$ and $\mathbf{1} = -\infty$.

We suppose that I is endowed with a metric d compatible with $\limsup$ and $\liminf$, i.e., $\limsup x_n = x$ and $\liminf x_n = x$ imply $\lim_{n\to\infty} d(x_n, x) = 0$, and which satisfies at least one of the following conditions:

$$d(x \oplus y,\ x' \oplus y') \leq d(x, x') + d(y, y') \tag{1}$$

$$d(x \oplus y,\ x' \oplus y') \leq \max\{d(x, x'),\ d(y, y')\}. \tag{2}$$

Both conditions (1) and (2) imply that :

$$d(x_n, y_n) \to 0 \quad \text{ implies } \quad d(x_n \oplus z,\ y_n \oplus z) \to 0.$$

We suppose further the monotonicity of the metric d, i.e.

$$x \preceq z \preceq y \ \Rightarrow\ d(x, y) \geq \max(d(x, z), d(y, z)). \tag{3}$$

For the case I) a) (i) on the interval $(-\infty, +\infty]$ introduce a metric

$$d(x, y) = |e^{-x} - e^{-y}|. \tag{4}$$

For the case II) on the interval $[a, b]$ introduce a metric

$$d(x, y) = |g(x) - g(y)|. \tag{5}$$

For the case III) b) on the interval $[-\infty, +\infty]$ introduce a metric

$$d(x, y) = \frac{2}{\pi}|\mathrm{arctg} x - \mathrm{arctg} y|. \tag{6}$$

2 The pseudo-probability

Let $(I, \oplus, \odot)$ be a semiring. Let Ω be a non-empty set. Let Σ be a $\sigma-$algebra of subsets of Ω.

In [4], the pseudo-integral of a bounded measurable function (for decomposable measure m) $f : \Omega \to I$ is defined. For the case II), the pseudo-integral reduces on $g-$integral, i.e.: $\int_\Omega^\oplus f \odot dm = g^{-1}(\int_\Omega g(f(x))dx)$.

Definition 3 *Let* $\sum$ *be* σ *- algebra of subsets of a set* Ω. **Pseudo - probability** *is a function* $\mathbf{P} : \sum \to I$ *with the properties*

(a) $\mathbf{P}(\emptyset) = \mathbf{0}$ *and* $\mathbf{P}(\Omega) = \mathbf{1}$,

(b) $\mathbf{P}(A \cup B) = \mathbf{P}(A) \oplus \mathbf{P}(B)\,, \quad A, B \in \sum, \quad A \cap B = \emptyset,$

(c) $A_i \in \sum,\ i \in N, \quad A_i \subseteq A_{i+1},\ i \in N \quad \Rightarrow \quad \lim\limits_{i \to \infty} \mathbf{P}(A_i) = \mathbf{P}\left(\bigcup\limits_{i=1}^{\infty} A_i\right).$

The triple $(\Omega, \Sigma, \mathbf{P})$ *is* **pseudo-probability space**.

An equivalent definition of pseudo - probability is obtained if the conditions (b) and (c) are replaced by the condition: $\mathbf{P}\left(\bigcup\limits_{i=1}^{\infty} A_i\right) = \bigoplus\limits_{i=1}^{\infty} \mathbf{P}(A_i)\,,$ where $\{A_i\}_{i \in N}$ is a sequence of pairwise disjoint sets from $\sum$ ($\sigma - \oplus -$ additivity of function $\mathbf{P}$). Let as notice that the pseudo-probability is specially case of decomposable measure.

In the case II), we have $\mathbf{P}(A) = g^{-1}(p(A))$, where p is usual probability. Then we say that $\mathbf{P}$ is distorted probability (see [1]).

Definition 4 *The function* $X : \Omega \to I$ *is* **pseudo-variable** *if*

$$X^{-1}((\cdot, x)) = \{\omega \in \Omega : \ X(\omega) \prec x\} = \{X \prec x\} \in \Sigma, \quad \text{for all } x \in I.$$

We also define the **distribution function** F *of pseudo-variable* X*, as*

$$F_X(x) = \mathbf{P}(\{X \prec x\}) = \mathbf{P}(\{\omega \in \Omega : X(\omega) \prec x\}).$$

Let $\sigma(I)$ be a minimal $\sigma-$algebra containing open balls in separable metric space (I, d). Let m be decomposable measure defined in measurable space $(I, \sigma(I))$.

Definition 5 *If there exists a function* ϕ_X *that holds*

$$F_X(x) = \int\limits_{X^{-1}((\cdot,x))}^{\oplus} \phi_X \odot d\mathbf{P}$$

then we say that X *is continuous pseudo-variable and that* ϕ_X *is the* **density function**.

Definition 6 *A pseudo-variable* X *is called* **integrable** *if there exists*

$$\mathbf{E}(X) = \int\limits_{\Omega}^{\oplus} x \odot \phi_X \odot d\mathbf{P}.$$

and then $E(X)$ *is* **the pseudo-expectation** *of the pseudo-variable* X.

In the case II) is $\mathbf{E}(X) = g^{-1}\left(\int\limits_0^{\infty} g\left(x\right) \cdot g\left(\phi_X\left(x\right)\right) dx\right)$.

Definition 7 *Continuous pseudo-variables X and Y are independent if holds*

$$\phi_{X,Y}(x,y) = \phi_X(x) \odot \phi_Y(y).$$

Definition 8 *The sequence $\{X_n\}$ of pseudo-variables* **converges in the pseudo-probability P** *towards X, denoted $X_n \xrightarrow{\mathbf{P}} X$, if for all $\varepsilon > 0$ we have*

$$\mathbf{P}(\{\omega \in \Omega : d(X_n(\omega), X(\omega)) \geq \varepsilon\}) \to \mathbf{0}.$$

Definition 9 *The sequence $\{X_n\}$ of pseudo-variables* **converges almost surely** *towards X, denoted $X_n \xrightarrow{a.s.} X$ if we have*

$$\mathbf{P}(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}) = \mathbf{1},$$

i.e.

$$\mathbf{P}(\{\omega \in \Omega : X_n(\omega) \nrightarrow X(\omega)\}) = \mathbf{0}.$$

In the idempotent cases I) and III), we have (see [5]):

Theorem 1 *Let X_n and X denote pseudo variables. Then $X_n \xrightarrow{\mathbf{P}} X$ implies $X_n \xrightarrow{a.s.} X$.*

Proof: We use $\{\omega : X_n(\omega) \nrightarrow X(\omega), n \to +\infty\} = \bigcup\limits_{k>0} \lim\limits_{m} \downarrow \bigcup\limits_{n \geq m} \{\omega \in \Omega : d(X_n(\omega), X(\omega)) \geq \frac{1}{k}\}$. Using the properties of an idempotent measure, we have

$$\begin{aligned}
& \mathbf{P}(\{\omega : X_n(\omega) \nrightarrow X(\omega), n \to +\infty\}) && (7)\\
= & \bigoplus_{k>0} \mathbf{P}(\lim_m \downarrow \bigcup_{n \geq m} \{\omega : d(X_n(\omega), X(\omega)) \geq \frac{1}{k}\}) && (8)\\
\preceq & \bigoplus_{k>0} \lim_m \downarrow \mathbf{P}(\bigcup_{n \geq m} \{\omega : d(X_n(\omega), X(\omega)) \geq \frac{1}{k}\}) && (9)\\
= & \bigoplus_{k>0} \lim_m \downarrow \bigoplus_{n \geq m} \mathbf{P}(\{\omega : d(X_n(\omega), X(\omega)) \geq \frac{1}{k}\}). && (10)
\end{aligned}$$

So, if $\mathbf{P}(\{\omega : d(X_n(\omega), X(\omega)) \geq \frac{1}{k}\}) \to \mathbf{0}$, $n \to +\infty$, then

$$\lim_{m} \downarrow \bigoplus_{n\geq m} \mathbf{P}(\{\omega : d(X_n(\omega), X(\omega)) \geq \tfrac{1}{k}\}) =$$
$$= \limsup_{n\to+\infty} \mathbf{P}(\{\omega : d(X_n(\omega), X(\omega)) \geq \tfrac{1}{k}\}) = \mathbf{0}$$
and $\mathbf{P}(\{\omega : X_n(\omega) \nrightarrow X(\omega), n \to +\infty\}) = \mathbf{0}$. □

Remark: The differences encountered with the classical probability theory were:

-(8) and (10) are equalities instead of inequalities ($\preceq$) because $\oplus$ is idempotent, i.e. for any sequence of sets $A_1, A_2, A_3, \ldots$: $\mathbf{P}(A_1 \cup A_2) = \mathbf{P}((A_1 \setminus A_2) \cup (A_1 \cap A_2) \cup (A_2 \setminus A_1)) = \mathbf{P}(A_1 \setminus A_2) \oplus \mathbf{P}(A_1 \cap A_2) \oplus \mathbf{P}(A_2 \setminus A_1) = \mathbf{P}(A_1 \setminus A_2) \oplus (\mathbf{P}(A_1 \cap A_2) \oplus \mathbf{P}(A_1 \cap A_2)) \oplus \mathbf{P}(A_2 \setminus A_1) = (\mathbf{P}(A_1 \setminus A_2) \oplus \mathbf{P}(A_1 \cap A_2)) \oplus (\mathbf{P}(A_1 \cap A_2) \oplus \mathbf{P}(A_2 \setminus A_1)) = \mathbf{P}(A_1) \oplus \mathbf{P}(A_2)$, and hence $\mathbf{P}(A_1 \cup \ldots \cup A_n) = \mathbf{P}(A_1) \oplus \ldots \oplus \mathbf{P}(A_n)$, and hence for $A_n = \{\omega : d(X_n(\omega), X(\omega)) \geq \frac{1}{k}\}$ and $B_k = \bigcup_{n=m}^{k} A_n$ (where $B_k \subseteq B_{k+1}$ so we can use property (c) of pseudo-probability): $\mathbf{P}(\bigcup_{n=m}^{\infty} A_n) = \mathbf{P}(\bigcup_{k=m}^{\infty} B_k) = \lim_{k\to\infty} \mathbf{P}(B_k) = \lim_{k\to\infty} \mathbf{P}(\bigcup_{n=m}^{k} A_n) = \lim_{k\to\infty} \bigoplus_{n=m}^{k} \mathbf{P}(A_n) = \bigoplus_{n=m}^{\infty} \mathbf{P}(A_n)$

-(9) is in general an inequality instead of an equality because of the non continuity of idempotent measures over nonincreasing sequences.

In the not idempotent case II), we have:

Theorem 2 *Let $\oplus$ be generated by g. Let X_n and X denote pseudo variables. Then $X_n \overset{a.s.}{\to} X$ implies $X_n \overset{\mathbf{P}}{\to} X$.*

Proof:
From $\mathbf{P}(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}) = \mathbf{1}$, i.e. $g^{-1}(p(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\})) = \mathbf{1}$, we obtain $p(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}) = 1$, because $g(\mathbf{1}) = 1$. In metric space (I, d), $X_n(\omega) \to X(\omega)$ is equivalent with $(\forall\delta > 0)(\exists n_0 \in N)(\forall n \in N) n \geq n_0 \Rightarrow \delta > d(X_n(\omega), X(\omega)) = |g(X_n(\omega)) - g(X(\omega))|$,
i.e. the sequence $\{g(X_n)\}$ of random variables converges almost surely towards $g(X)$. In usual probability theory, almost surely convergence implies convergence in the probability, so we have that for all $\varepsilon > 0$ hold:

$p(\{\omega \in \Omega : |g(X_n(\omega)) - g(X(\omega))| \geq \varepsilon\}) \to 0, \quad n \to \infty.$

Finally, from

$\mathbf{P}(\{\omega \in \Omega : d(X_n(\omega), X(\omega)) \geq \varepsilon\}) \to \mathbf{0} \Leftrightarrow$

$\Leftrightarrow 0 = \lim_{n\to\infty} d(\mathbf{P}(\{\omega \in \Omega : d(X_n(\omega), X(\omega)) \geq \varepsilon\}), \mathbf{0}) =$

$= \lim\limits_{n\to\infty} |g(\mathbf{P}(\{\omega \in \Omega : d(X_n(\omega), X(\omega)) \geq \varepsilon\})) - g(\mathbf{0})| =$
$= \lim\limits_{n\to\infty} p(\{\omega \in \Omega : d(X_n(\omega), X(\omega)) \geq \varepsilon\}) =$
$= \lim\limits_{n\to\infty} p(\{\omega \in \Omega : |g(X_n(\omega)) - g(X(\omega))| \geq \varepsilon\})$

we obtain that the sequence $\{X_n\}$ of pseudo-variables converges in the pseudo-probability $\mathbf{P}$ towards X. □

3 The law of large numbers

Let g be the continuous strictly monotonic function. Then, we say for

$$S_n(x_1, x_2, ..., x_n) = g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right), \quad n \in N$$

that they are "quasi-arithmetic means".

In special cases, we have:

1) $g(x) = x$, $S_n(x_1, x_2, ..., x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$ (arithmetic mean),

2) $g(x) = x^2$, $S_n(x_1, x_2, ..., x_n) = [\frac{1}{n}\sum_{i=1}^{n} x_i^2]^{1/2}$ (quadratic mean),

3) $g(x) = x^\alpha$, $S_n(x_1, x_2, ..., x_n) = [\frac{1}{n}\sum_{i=1}^{n} x_i^\alpha]^{1/\alpha}$ (root-power mean),

4) $g(x) = x^{-1}$, $S_n(x_1, x_2, ..., x_n) = [\frac{1}{n}\sum_{i=1}^{n} \frac{1}{x_i}]^{-1}$, (harmonic mean),

5) $g(x) = \log x$, $S_n(x_1, x_2, ..., x_n) = [\prod_{i=1}^{n} x_i]^{1/n}$, (geometric mean),

6) $g(x) = e^{\alpha x}$, $S_n(x_1, x_2, ..., x_n) = \frac{1}{\alpha}\ln[\frac{1}{n}\sum_{i=1}^{n} e^{\alpha x_i}]$ (exponential mean).

The following theorem hold (see [6]):

Theorem 3 *Let $X_1, X_2, ...$ be a sequence of independent integrable pseudo-variables identically distributed,* $\mathbf{E}(X_n) = a, n = 1, 2, ...$ *. Then* $S_n \xrightarrow{\mathbf{P}} a$.

Proof. We prove that the following holds: $\lim\limits_{n\to\infty} \mathbf{P}(\{d(S_n, a) \geq \varepsilon\}) = \mathbf{0}$, for all $\varepsilon > 0$.

$d(\mathbf{P}(\{d(S_n, a) \geq \varepsilon\}), \mathbf{0}) = |g(\mathbf{P}(\{d(S_n, a) \geq \varepsilon\})) - g(\mathbf{0})| =$
$= |p(\{d(S_n, a) \geq \varepsilon\})) - 0| = p(\{d(S_n, a) \geq \varepsilon\})) = p(\{|g(S_n) - g(a)| \geq \varepsilon\}) =$
$= p(\{|g(S_n) - g(a)| \geq \varepsilon\}) = p(\{|\frac{1}{n}\sum_{i=1}^{n} g(X_i) - g(a)| \geq \varepsilon\}).$

As the variables $Y_i = g(X_i)$, $i = 1, ..., n$ satisfy the usual weak law of large numbers, this statement follows, i.e. $p(|\frac{1}{n}\sum_{i=1}^{n} g(X_i) - g(a)| \geq \varepsilon\}) \rightarrow 0$. $\square$

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