Some properties of pseudo-measures and pseudo-probability

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***Abstract*—This paper is based on the concept and results from the pseudo-probability theory. Since the classical probability is the mapping from some σ-algebra into the $(\mathbb{R}_0^+, +, \cdot)$, the pseudo-probability space is obtained by substituting $(\mathbb{R}_0^+, +, \cdot)$ with more general structure $(I, \oplus, \odot)$ called semiring. Set I is usually some subinterval of $[-\infty, \infty]$, while operations $\oplus$ and $\odot$ are generalizations of the classical operations, known as pseudo-addition and pseudo-multiplication, respectively. In this paper some properties of the pseudo-probability measures and corresponding integrals constructed in the given setting are investigated.**

I. Introduction

Results presented in this paper belong to the area of pseudo-measures, pseudo-integrals and pseudo-probability. Pseudo-probability is considered to be a generalization of usual probability (see [1]). In this article we presented basic properties of pseudo-probability space, and some properties of $\sup$-measure and $\inf$-measure and corresponding integrals.

Preliminary notions of pseudo-analysis (such as pseudo-operations, semiring, pseudo-integral) are given in Section II ([3], [4], [5], [9], [10]). The Section III contains definitions of pseudo-probability, pseudo-random variable and their distribution function. Convergence of sequence of pseudo-random variables (convergence in the pseudo-probability and almost surely convergence) is also presented in this section ([7], [12], [13], [14]). In Section IV we study some properties of $\sup$-integral ([8]) and $\inf$-integral.

II. Preliminaries

In this section, we shall briefly present some notions from the pseudo-analysis, see [4], [5], [9], [10].

Let $\preceq$ be an order defined on a nonempty set I, and let $\emptyset \neq I^* \subset I$.

Definition 1: The **pseudo-operation** on the set I is a binary operation $* : I \times I \to I$ which is commutative, associative, "positively non-decreasing" - in the sense that for all $u \in I^*$, $x \preceq y$ implies $x * u \preceq y * u$, and for which there exists a neutral element $e \in I$.

The element $u \in I$ is a **null element** of the operation $* : I^2 \to I$ if for any $x \in I$, $x * u = u * x = u$ holds.

Definition 2: Let $\oplus$ and $\odot$ be two pseudo-operations defined on the ordered set $(I, \preceq)$, with $\mathbf{0}$ and $\mathbf{1}$ as neutral elements, respectively. Let $I^{\oplus} = I$, for the first operation, and $I^{\odot} = \{x \in I \mid \mathbf{0} \preceq x\}$, for the second operation. If $\odot$ is a distributive operation with respect to pseudo-operation $\oplus$ and $\mathbf{0}$ is a null element of the operation $\odot$, than the triplet $(I, \oplus, \odot)$ is called a **semiring**.
The semiring $(I, \oplus, \odot)$ will be denoted by $I^{\oplus, \odot}$.

Let I be a subinterval of $[-\infty, \infty]$ (we will usually take closed subinterval, but in some cases semiclosed subinterval are taken). Then the operations $\oplus$ and $\odot$ are called **pseudo-addition** and **pseudo-multiplication**.

Pseudo-operation $*$ is **idempotent** if for any $x \in I$, $x*x = x$ holds.

There are three important types of real semirings ([10]).

(I) The $\oplus$ is idempotent operation ($\oplus = \sup$ or $\oplus = \inf$), and $\odot$ is not idempotent operation.
Example 1:
1) $x \oplus y = \min(x, y), \quad x \odot y = x + y$,
on the interval $(-\infty, \infty]$ ordered by $\geq$, with neutral elements $\mathbf{0} = \infty$ and $\mathbf{1} = 0$. We denote this semiring by $(-\infty, \infty]^{\min,+}$.
2) $x \oplus y = \max(x, y), \quad x \odot y = x + y$,
on the interval $[-\infty, \infty)$ ordered by $\leq$, with neutral elements $\mathbf{0} = -\infty$ and $\mathbf{1} = 0$.
3) $x \oplus y = \min(x, y), \quad x \odot y = x \cdot y$,
on the interval $(0, \infty]$ ordered by $\geq$, with neutral elements $\mathbf{0} = \infty$ and $\mathbf{1} = 1$.
4) $x \oplus y = \max(x, y), \quad x \odot y = x \cdot y$,
on the interval $[0, \infty)$ ordered by $\leq$, with neutral elements $\mathbf{0} = 0$ and $\mathbf{1} = 1$.

(II) Both of operations $\oplus$ and $\odot$ are generated by strictly monotone and continuous function $g : [a, b] \to [0, \infty]$ in the following sense:
$x \oplus y = g^{-1}(g(x) + g(y))$,
$x \odot y = g^{-1}(g(x) \cdot g(y))$,
with the convention $0 \cdot (+\infty) = 0$ (see [10]). In this case, for neutral elements we have that $g(\mathbf{0}) = 0$ and $g(\mathbf{1}) = 1$, i.e. $\mathbf{0} = a$ or $\mathbf{0} = b$. The order is defined as $x \preceq y \Leftrightarrow g(x) \leq g(y)$.

(III) Both of $\oplus$ and $\odot$ are idempotent, i.e. $([a,b],\oplus,\odot) = ([a,b],\sup,\inf)$, or $([a,b],\oplus,\odot) = ([a,b],\inf,\sup)$.

Example 2:

1) $([a,b],\oplus,\odot) = ([-\infty,\infty],\max,\min)$, with $\mathbf{0} = -\infty$ and $\mathbf{1} = \infty$.
2) $([a,b],\oplus,\odot) = ([-\infty,\infty],\min,\max)$, with $\mathbf{0} = \infty$ and $\mathbf{1} = -\infty$.

With purpose of defining various types of convergence in semiring $I^{\oplus,\odot}$, we have to introduce a metric d (see [10]) compatible with pseudo-operations $\oplus$ and $\odot$, in the sense that $\limsup_{n\to\infty} x_n = x$ and $\liminf_{n\to\infty} x_n = x$ imply $\lim_{n\to\infty} d(x_n, x) = 0$, and which satisfies at least one of the following conditions:

$$d(x \oplus y,\ x' \oplus y') \leq d(x,x') + d(y,y') \tag{1}$$

$$d(x \oplus y,\ x' \oplus y') \leq \max\{d(x,x'), d(y,y')\}. \tag{2}$$

Both conditions (1) and (2) implies that

$$\lim_{n\to\infty} d(x_n,y_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} d(x_n \oplus z,\ y_n \oplus z) = 0.$$

We suppose further the monotonicity of the metric d, i.e. that

$$x \preceq z \preceq y \quad \Rightarrow \quad d(x,y) \geq \max(d(x,z), d(y,z)). \tag{3}$$

Example 3:

1) For the semiring $((-\infty,+\infty],\min,+)$ in example 1 (type (I)) we can consider a metric
$$d(x,y) = |e^{-x} - e^{-y}|. \tag{4}$$
2) For the semiring of type (II) on the interval $[a,b]$ we consider a metric
$$d(x,y) = |g(x) - g(y)|. \tag{5}$$
3) For the $([-\infty,+\infty],\min,\max)$ in example 2 (type (III)) we consider a metric
$$d(x,y) = \frac{2}{\pi}|\operatorname{arctg} x - \operatorname{arctg} y|. \tag{6}$$

Let $([a,b],\oplus,\odot)$ be a semiring. For σ-algebra Σ on Ω, function $m : \Sigma \to [a,b]$ is a **σ-$\oplus$-decomposable measure** on (Ω,Σ) if (see [10]):

(a) $m(\emptyset) = \mathbf{0}$,

(b) $m(\bigcup_{i\in\mathbb{N}} A_i) = \bigoplus_{i\in\mathbb{N}} m(A_i)$ for any collection of (pairwise disjoint if $\oplus$ is not idempotent) sets $A_i \in \Sigma$, $i \in \mathbb{N}$.

Pseudo-probability P is a σ-$\oplus$-decomposable measure on (Ω,Σ) with the additional property $\mathsf{P}(\Omega) = \mathbf{1}$. The triple $(\Omega,\Sigma,\mathsf{P})$ is **pseudo-probability space**.

For the sake of the completeness, some properties of σ-$\oplus$-measures and appropriate integrals necessary for further consideration will be given (see [10], Chapters 2 and 8, and [9]). Let m be an σ-$\oplus$-measure on Ω with values in the semiring $([0,\infty],\oplus,\odot)$, such that $m(\Omega) < \infty$. We adopt the convention $\infty \cdot 0 = 0$. Let $\chi_A : \Omega \to [0,\infty]$ denotes **pseudo-characteristic function** of a set $A \subset \Omega$, i.e.

$$\chi_A(x) = \begin{cases} \mathbf{0} & , \quad x \notin A \\ \mathbf{1} & , \quad x \in A \end{cases}.$$

The **pseudo-integral** (see [10]) of a elementary function $e = \bigoplus_{i=1}^{\infty} a_i \odot \chi_{A_i}$ for $a_i \in [0,\infty)$ and sets A_i (disjoint for nonidempotent $\oplus$), is defined by

$$\int_\Omega^\oplus e \odot dm = \bigoplus_{i=1}^{\infty} a_i \odot m(A_i).$$

The **pseudo-integral** of a bounded measurable (from below for idempotent $\oplus$) function $f : \Omega \to [0,\infty)$, for which, if $\oplus$ is not idempotent for each $\varepsilon > 0$, there exists a monotone ε-net in $f(\Omega)$, is defined by (see [10])

$$\int_\Omega^\oplus f \odot dm = \lim_{n\to\infty} \int_\Omega^\oplus \varphi_n(x) \odot dm,$$

where φ_n, $n \in \mathbb{N}$ is the sequence of elementary functions. Sequence φ_n, $n \in \mathbb{N}$ exists, and the definition of pseudo-integral is independent of choice of φ_n, $n \in \mathbb{N}$ (see Theorems 8.1 and 8.2 in [10]).

Remark 1: Function f is measurable from below if for any $c \in [0,\infty]$ the sets $\{x \mid f(x) \leq c\}$ and $\{x \mid f(x) < c\}$ are measurable.

Remark 2: For the case (II), the pseudo-integral is the $g-$integral (see [9], [10]), i.e.

$$\int_{[c,d]}^\oplus f \odot dm = g^{-1}\Big(\int_{[c,d]} g \circ f d(g \circ m)\Big),$$

and for pseudo-probability P we have that $\mathsf{P}(A) = g^{-1}(p(A))$, where p is usual probability. Function P is called a distorted probability (see [2]).

The following theorem is proved in [10].

Theorem 1: Let m be a σ-$\oplus$-decomposable measure. For any $c \in [0,\infty]$, any measurable $A \subseteq \Omega$, and any bounded measurable, or measurable from below for idempotent $\oplus$, functions $f : \Omega \to [0,\infty)$ and $g : \Omega \to [0,\infty)$ holds

1) $m(A) = \int_\Omega^\oplus \chi_A(x) \odot dm$,
2) $\int_\Omega^\oplus (c \odot f) \odot dm = c \odot \int_\Omega^\oplus f \odot dm$,
3) $\int_\Omega^\oplus (f \oplus g) \odot dm = \int_\Omega^\oplus f \odot dm \oplus \int_\Omega^\oplus g \odot dm$,
4) $f \preceq g \quad \Rightarrow \quad \int_\Omega^\oplus f \odot dm \preceq \int_\Omega^\oplus g \odot dm$.

III. Convergence of Sequence of Pseudo-Random Variables

Results presented in this section can be found in [7], [12], [13], [14].

Let $(\Omega, \Sigma, \mathsf{P})$ be a pseudo-probability space based od semiring $(I, \oplus, \odot)$.

Definition 3: The function $X : \Omega \to I$ is a **pseudo-random variable** if

$$X^{-1}((\cdot, x)) = \{\omega \in \Omega \mid X(\omega) \prec x\} = \{X \prec x\} \in \Sigma$$

for all $x \in I$.

We also define the **distribution function** F of pseudo-random variable X, as

$$F_X(x) = \mathsf{P}(\{X \prec x\}).$$

Let $\sigma(I)$ be a minimal $\sigma-$algebra containing open balls in separable metric space (I, d) (with metric d compatible with $\oplus$ and $\odot$). Let m be a decomposable measure defined in the measurable space $(I, \sigma(I))$.

If there exists a function ϕ satisfying

$$F_X(x) = \int\limits_{X^{-1}((\cdot,x))}^{\oplus} \phi_X \odot dm$$

then ϕ is called a **density function**.

Various types of convergence of sequence of pseudo-random variables based on semiring can be introduced. In [12] we introduce "convergence in the pseudo-probability" and "almost surely convergence". The following relations between them are also stated in [12] (theorems 2 and 3).

Definition 4: The sequence $\{X_n\}_{n\in\mathbb{N}}$ of pseudo-random variables **converges in the pseudo-probability** P towards X, noted by $X_n \xrightarrow{\mathsf{P}} X$, if for all $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathsf{P}(\{\omega \in \Omega \mid d(X_n(\omega), X(\omega)) \geq \varepsilon\}) = \mathbf{0}.$$

Definition 5: The sequence $\{X_n\}_{n\in\mathbb{N}}$ of pseudo-random variables **converges almost surely** towards X, noted by $X_n \xrightarrow{a.s.} X$, if we have

$$\mathsf{P}(\{\omega \in \Omega \mid X_n(\omega) \to X(\omega)\}) = \mathbf{1},$$

i.e.

$$\mathsf{P}(\{\omega \in \Omega \mid X_n(\omega) \nrightarrow X(\omega)\}) = \mathbf{0}.$$

In the idempotent cases I) and III), we have (see [13]):

Theorem 2: Let X_n and X denote pseudo variables. Then $X_n \xrightarrow{\mathsf{P}} X$ implies $X_n \xrightarrow{a.s.} X$.

In the case of not idempotent type (II) of semiring, we have

Theorem 3: Let X_n and X denote pseudo variables. Then $X_n \xrightarrow{a.s.} X$ implies $X_n \xrightarrow{\mathsf{P}} X$.

In this section we also present the concept of "mean" value of pseudo-random variables. For sequence of independent, identically distributed pseudo-random variables X_n, $n \in \mathbb{N}$, Theorem 4 (see [13]) claim that, like in usual probability theory, quasi-arithmetic means of pseudo-random variables X_n converge "in probability". For more details see [13], [14], [12]).

The **pseudo-expectation** of the pseudo-random variable X is defined with

$$\mathsf{E}(X) = \int\limits_{\Omega}^{\oplus} x \odot \phi_X(x) \odot dm.$$

Example 4: In the case of pseudo-probability based on semiring $((-\infty, \infty], \min, +)$ from Example 1 (type (I)), the pseudo-expectation is

$$\mathsf{E}(X) = \inf_{x\in(-\infty,\infty]} \{x + \phi_X(x)\}.$$

Definition 6: The pseudo-random variables X and Y, with densities ϕ_X and ϕ_Y respectively, are **independent** if it holds

$$\phi_{X,Y}(x, y) = \phi_X(x) \odot \phi_Y(y).$$

Let g be a continuous, strictly monotonic function. Then

$$S_n(x_1, x_2, ..., x_n) = g^{-1}(\frac{1}{n}\sum_{i=1}^{n} g(x_i)), \quad n \in N$$

is called a **quasi-arithmetic mean**.

In some special cases, we have the table (see [12])

$g(x)$	$S_n(x_1, x_2, ..., x_n)$	means
x	$\frac{1}{n}\sum_{i=1}^{n} x_i$	arithmetic
x^2	$[\frac{1}{n}\sum_{i=1}^{n} x_i^2]^{1/2}$	quadratic
x^α	$[\frac{1}{n}\sum_{i=1}^{n} x_i^\alpha]^{1/\alpha}$	root-power
x^{-1}	$[\frac{1}{n}\sum_{i=1}^{n} \frac{1}{x_i}]^{-1}$	harmonic
$\log x$	$[\prod_{i=1}^{n} x_i]^{1/n}$	geometric
$e^{\alpha x}$	$\frac{1}{\alpha}\ln[\frac{1}{n}\sum_{i=1}^{n} e^{\alpha x_i}]$	exponential

We consider the semiring of type (II), in which the metric is defined with $d(x, y) = \|g(x) - g(y)\|$. The following theorem holds (see [13]):

Theorem 4: Let $X_1, X_2, ...$ be a sequence of independent pseudo-variables, identically distributed, with equal $\mathsf{E}(X_n) = a$, $n \in \mathbb{N}$. Then

$$S_n \xrightarrow{\mathsf{P}} a.$$

IV. Some additional properties of pseudo-measure and related integrals

In this section we will be considered pseudo-measures and pseudo-integrals with values in semirings $([0, \infty], \min, \odot)$ and $([0, \infty], \max, \odot)$, where $\odot$ is continuous operation.

Example 5: Let $\Omega = \mathbb{N}$, and let $\mathsf{P} : \mathcal{P}(\mathbb{N}) \to [0,\infty]$ be a function defined by

$$\mathsf{P}(\emptyset) = 0, \qquad \mathsf{P}(A) = 1 - \frac{1}{\sup A}, \quad \emptyset \neq A \subseteq \mathbb{N}.$$

Function P is a pseudo-probability on $\mathbb{N}$, with values in semiring $([0,\infty], \max, \odot)$.

Semiring $([0,\infty], \min, \odot)$ with continuous $\odot$ is a semiring of the type (I). It is known that $\inf$-measure m is given by some function $\varphi : \Omega \to [0,\infty]$ in the following way:

$$m(A) = \inf_{x\in A} \varphi(x), \quad A \subseteq \Omega.$$

Analogously, $\sup$-measure m (with value in semiring $([0,\infty], \max, \odot)$) is given by some function $\varphi : \Omega \to [0,\infty]$ in the following way:

$$m(A) = \sup_{x\in A} \varphi(x), \quad A \subseteq \Omega.$$

Function φ is called **density-function** of pseudo-measure m.

Therefore, pseudo-integrals based on $\min$ and $\max$ measures m respectively, i.e. with values in semirings $([0,\infty], \min, \odot)$ and $([0,\infty], \max, \odot)$ (with continuous operation $\odot$), can be represented in the form

$$\int_A^{\inf} f \odot dm = \inf_{\omega\in A} \left\{ f(\omega) \odot \varphi(\omega) \right\},$$

respectively

$$\int_A^{\sup} f \odot dm = \sup_{\omega\in A} \left\{ f(\omega) \odot \varphi(\omega) \right\},$$

where $A \subseteq \Omega$, φ is density-function of pseudo-measure m, and $f : A \to [0,\infty]$ is measurable function.

In [8] the following theorem was proved.

Theorem 5: Let $\odot$ be a continuous pseudo-multiplication. For the $\sup$-measure m and a family of functions $f_j : \Omega \to [0,\infty)$, $j \in J$, it holds

$$\int_\Omega^{\sup} (\sup_{j\in J} f_j) \odot dm = \sup_{j\in J} \int_\Omega^{\sup} f_j \odot dm.$$

Let us consider now the semiring $([0,\infty], \min, \odot)$ with continuous operation $\odot$ (type (I), see Example 1). Let φ be a density-function of $\inf$-measure m.

In this case we obtain the following result.

Theorem 6: For the $\inf$-decomposable measure m and a family of functions $f_j : \Omega \to (0,\infty]$, $j \in J$, it holds

$$\int_\Omega^{\inf} (\inf_{j\in J} f_j) \odot dm = \inf_{j\in J} \int_\Omega^{\inf} f_j \odot dm.$$

Proof: We have

$$\int_\Omega^{\inf} (\inf_{j\in J} f_j) \odot dm = \inf_{\omega\in\Omega} \left\{ (\inf_{j\in J} f_j)(\omega) \odot \varphi(\omega) \right\}$$

$$= \inf_{\omega\in\Omega} \left\{ (\inf_{j\in J} f_j(\omega)) \odot \varphi(\omega) \right\}$$

$$\overset{[*]}{=} \inf_{\omega\in\Omega} \inf_{j\in J} \left\{ f_j(\omega) \odot \varphi(\omega) \right\}$$

$$= \inf_{j\in J} \inf_{\omega\in\Omega} \left\{ f_j(\omega) \odot \varphi(\omega) \right\} = \inf_{j\in J} \int_\Omega^{\inf} f_j \odot dm.$$

[*] *Using continuity of* $\odot$. □

Considering various properties of the pseudo-integrals (see Theorem 1), we analyze some additional relations, i.e. we analyze under which conditions these relations hold.

Precisely, in this paper, we will analyze some consequences of the following property of continuous pseudo-multiplication $\odot$ generated by g (i.e. $x \odot y = g^{-1}(g(x) \cdot g(y))$), as second operation of semiring $I = ([0,\infty], \max, \odot)$ (semiring of the type (I)):

$$\forall a, b, c \in [0,\infty], \quad (a+b) \odot c \leq a \odot c + b \odot c, \qquad [\blacktriangle]$$

which is satisfied for a wide class of pseudo-operations $\odot$.

Let m be a $\sup$-decomposable measure determined by density-function $\varphi : \Omega \to [0,\infty]$, and let $I = ([0,\infty], \max, \odot)$ be a semiring, where the pseudo-multiplication $\odot$ is generated by strictly monotone and continuous function $g : [a,b] \to [0,\infty]$ (i.e. $x \odot y = g^{-1}(g(x) \cdot g(y))$).

Proposition 1: If the pseudo-multiplication $\odot$ has the property $[\blacktriangle]$, then for $f : \Omega \to [0,\infty]$ and $h : \Omega \to [0,\infty]$ holds

$$\int_\Omega^{\sup} (f+h) \odot dm \leq \int_\Omega^{\sup} f \odot dm + \int_\Omega^{\sup} h \odot dm \qquad (7)$$

Proof:

$$\int_\Omega^{\sup} (f+h) \odot dm = \sup_{\omega\in\Omega} \left\{ (f+h)(\omega) \odot \varphi(\omega) \right\}$$

$$= \sup_{\omega\in\Omega} \left\{ (f(\omega) + h(\omega)) \odot \varphi(\omega) \right\}$$

$$\overset{[\blacktriangle]}{\leq} \sup_{\omega\in\Omega} \left\{ f(\omega) \odot \varphi(\omega) + h(\omega) \odot \varphi(\omega) \right\}$$

$$\leq \sup_{\omega\in\Omega} \left\{ f(\omega) \odot \varphi(\omega) \right\} + \sup_{\omega\in\Omega} \left\{ h(\omega) \odot \varphi(\omega) \right\}$$

$$= \int_\Omega^{\sup} f \odot dm + \int_\Omega^{\sup} h \odot dm. \qquad \square$$

Proposition 2: If the pseudo-multiplication $\odot$ has the property $[\blacktriangle]$, then for $f : \Omega \to [0,\infty]$ and $h : \Omega \to [0,\infty]$ holds

$$\left| \int_{[0,\infty]}^{\sup} f \odot dm - \int_{[0,\infty]}^{\sup} h \odot dm \right| \leq \int_{[0,\infty]}^{\sup} |f-h| \odot dm. \qquad (8)$$

Proof: Suppose that $\int_\Omega^{\sup} f \odot dm \geq \int_\Omega^{\sup} h \odot dm$, i.e.

$$\left| \int_{[0,\infty]}^{\sup} f \odot dm - \int_{[0,\infty]}^{\sup} h \odot dm \right| = \int_{[0,\infty]}^{\sup} f \odot dm - \int_{[0,\infty]}^{\sup} h \odot dm$$

(the opposite case is analogous). Positive functions f and h satisfies $f(\omega) \leq |f-h|(\omega)+h(\omega)$. Density-function φ is also positive, so that $f(\omega) \odot \varphi(\omega) \leq (|f-h|(\omega)+h(\omega)) \odot \varphi(\omega)$. Hence, using Proposition 1, we obtain

$$\int_{\Omega}^{\sup} f \odot dm = \sup_{\omega\in\Omega} \{f(\omega) \odot \varphi(\omega)\}$$
$$\leq \sup_{\omega\in\Omega} \{(|f-h|(\omega)+h(\omega)) \odot \varphi(\omega)\}$$
$$= \int_{\Omega}^{\sup} (|f-h|+h) \odot dm$$
$$\leq \int_{\Omega}^{\sup} |f-h| \odot dm + \int_{\Omega}^{\sup} h \odot dm,$$

so that

$$\left| \int_{[0,\infty]}^{\sup} f \odot dm - \int_{[0,\infty]}^{\sup} h \odot dm \right|$$
$$= \int_{[0,\infty]}^{\sup} f \odot dm - \int_{[0,\infty]}^{\sup} h \odot dm$$
$$\leq \int_{[0,\infty]}^{\sup} |f-h| \odot dm. \qquad \square$$

V. CONCLUSION

Results given in this paper present one more contribution in the field of non-additive measures, pseudo-integrals and pseudo-probabilities. We will use these results for further research in these areas, to obtain more significant properties of pseudo-integrals. Our investigation will be focused on analyzing thoose properties of pseudo-operations which provide the claims of Propositoins 1 and 2, and other useful results.

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