Large deviation convergence of generated pseudo measures

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Abstract: The large deviation theory is a tool for asymptotic computation of very small probabilities. It is used for study of the convergence of "very small" probabilities. We shall give an short overwiev of some approaches of large deviation theory, and we shall consider the large deviation convergence of a sequence of generated pseudo-measures to sup*-decomposable measure.*

Key words and phrases: Large deviation principle, Idempotent probability, Semiring, Decomposable measure, g-integral.

1 Introduction

Much of the credit for the modern theory of large deviations and its various applications goes to S.R.S. Varadhan, Donsker, Freidlin i Wentzell (see [1] and [2]). This theory has found many applications in information theory, coding theory, image processing, statistical mechanics, finite state Markov chains, etc. (see [1], [2], [9]). The basic approach is based on the probability theory. The purpose of large deviation theory is characterization of the limit behavior of family of probability measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \to 0$.

Let $\mathcal{X}$ be a topological space (for example, Polish space), let $\mathcal{B}_\mathcal{X}$ be a completed Borel σ-algebra on $\mathcal{X}$, and let $\{\mathsf{P}_\varepsilon\}_{\varepsilon>0}$ be a family of probability measures on $(\mathcal{X}, \mathcal{B}_\mathcal{X})$. The family $\{\mathsf{P}_\varepsilon\}_{\varepsilon>0}$ satisfies the *large deviation principle* (LDP for short) with a *rate function* $I : \mathcal{X} \to [0, \infty]$ (i.e. lower semicontinuous function I) if for all $A \in \mathcal{B}_\mathcal{X}$

$$-\inf_{x\in \mathring{A}} I(x) \leq \liminf_{\varepsilon\to 0^+} \varepsilon \ln \mathsf{P}_\varepsilon(A) \leq \limsup_{\varepsilon\to 0^+} \varepsilon \ln \mathsf{P}_\varepsilon(A) \leq -\inf_{x\in \overline{A}} I(x), \qquad (1)$$

where $\mathring{A}$ and $\overline{A}$ are interior and closure of A respectively. If the rate function exists, it is uniquely determined. Main tasks are:

1. establishing necessary and/or sufficient conditions for convergence, i.e. for existence of rate function,
2. developing technics for computing the rate function.

Theorems of Sanov, Cramér, Gärtner-Ellis, etc. present the basic results in the theory of LDP (see [1] and [2]). In the section 2 we repeat, for the sake of completeness, the basic definition of the large deviation convergence for the family of usual probabilities and idempotent probability as the limit, and the theorem of Portmanteau. In the section 3 we present the convergence results from [4] and three theorems on sup-integral related to the results of [7]. In the section 4 we introduce a large deviation principle for the sequence of $\oplus$-measures; also, we give one characterization of this large deviation convergence.

2 Preliminaries

The theory of non-additive measures (see [3], [6], [7]) also take a part in the investigation of the large deviation convergence. One of the approaches is studying the convergence of the family of usual probability measures to idempotent sup-measure as it is described bellow (see [7]). Let $\mathcal{X}$ be a Tihonov topological space with Borel σ-algebra $\mathcal{B}_{\mathcal{X}}$. Let we denote $\mathbb{R}^+ = [0, \infty)$, let $\mathcal{F}$ be a collection of closed subsets of $\mathcal{X}$, and let $C_b^+(\mathcal{X})$, $\overline{C}_b^+(\mathcal{X})$ and $\underline{C}_b^+(\mathcal{X})$ denote the respective sets of continuous, bounded $\mathbb{R}^+$-valued functions on $\mathcal{X}$, upper semi-continuous, bounded $\mathbb{R}^+$-valued functions on $\mathcal{X}$, and lower semi-continuous, bounded $\mathbb{R}^+$-valued functions on $\mathcal{X}$,

Let Φ be a directed set, let P_ϕ, $\phi \in \Phi$ be a net (i.e. generalized sequence) of probability measures on $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$, let r_ϕ, $\phi \in \Phi$ be a net of real numbers with properties $r_\phi > 1$, $\phi \in \Phi$ and $\lim\limits_{\phi \in \Phi} r_\phi = \infty$. Finally, let Π be an $\mathcal{F}$-idempotent probability measure on $\mathcal{X}$. Denote $\|f\|_\phi = \left(\int\limits_{\mathcal{X}} f^{r_\phi} d\mathsf{P}_\phi \right)^{1/r_\phi}$.

Definition 1 *The net* P_ϕ, $\phi \in \Phi$ large deviation converge at rate r_ϕ, $\phi \in \Phi$ *to* Π *(*LD *converge, for short) if for all* $f \in C_b^+(\mathcal{X})$

$$\lim_{\phi \in \Phi} \|f\|_\phi = \bigvee_{\mathcal{X}} f d\Pi.$$

If the limit value exists, it is uniquely determined. The theorem of Portmanteau (see [7]) establishes some equivalent statements for LD convergence.

Theorem 1 (Portmanteau) *The following conditions are equivalent:*

(1) P_ϕ, $\phi \in \Phi$ LD *converge at rate* r_ϕ, $\phi \in \Phi$ *to* Π.

(2) (2.a) $\forall g \in \underline{C}_b^+(\mathcal{X}),\ \liminf\limits_{\phi \in \Phi} \|g\|_\phi \geq \bigvee\limits_{\mathcal{X}} g d\Pi,$

(2.b) $\forall f \in \overline{C}_b^+(\mathcal{X}),\ \limsup_{\phi\in\Phi} \|f\|_\phi \le \bigvee_{\mathcal{X}} f d\Pi.$

(3) (3.a) *for any open set $G \subseteq \mathcal{X}$ is satisfied*

$$\liminf_{\phi\in\Phi} \mathsf{P}_\phi^{1/r_\phi}(G) \ge \Pi(G),$$

(3.b) *for any closed set $F \subseteq \mathcal{X}$ is satisfied*

$$\limsup_{\phi\in\Phi} \mathsf{P}_\phi^{1/r_\phi}(F) \le \Pi(F).$$

3 Pseudo-operations and convergence of generated measures and integrals

In this section we shall present some pseudo-analysis notions and results (see [3], [6] and [4]). Let $[a,b] \subseteq [-\infty,\infty]$ (in some cases semiclosed subintervals are taken), $\oplus$ and $\odot$ binary operations on $[a,b]$, and let $\preceq$ be a total order on $[a,b]$.

Definition 2 *The triple* $([a,b],\oplus,\odot)$ *is called* semiring *if*

(a) $\oplus$ *(*pseudo-addition*) is associative, commutative, nondecreasing w.r.t.* $\preceq$ *operation with neutral (*zero*) element* $\mathbf{0}$ *(usually,* $\mathbf{0}$ *is either* a *or* b*),*

(b) $\odot$ *(*pseudo-multiplication*) is associative, commutative, positively nondecreasing w.r.t.* $\preceq$ *operation* $(\forall x,y,z \in [a,b], (x \preceq y \wedge \mathbf{0} \preceq z) \Rightarrow x\odot z \preceq y \odot z)$ *with neutral (*unit*) element* $\mathbf{1}$*,*

(c) (c.1) $\forall x \in [a,b], \mathbf{0}\odot x = \mathbf{0}$*,*

(c.2) $\odot$ *is distributive w.r.t.* $\oplus$*.*

There are three important types of semirings.

(I) The $\oplus$ is idempotent operation ($\oplus = \sup$ or $\oplus = \inf$), and $\odot$ is not,

(II) both of $\oplus$ and $\odot$ are generated by strictly monotone and continuous function $g : [a,b] \to [0,\infty]$ in the following sense:

$x \oplus y = g^{-1}(g(x)+g(y)), \qquad x\odot y = g^{-1}(g(x)\cdot g(y)),$

(in this case we have $g(\mathbf{0}) = 0$ and $g(\mathbf{1}) = 1$)

(III) both of $\oplus$ and $\odot$ are idempotent $(([a,b],\oplus,\odot) = ([a,b],\sup,\inf)$ or $([a,b],\oplus,\odot) = ([a,b],\inf,\sup))$.

In the case II, by Aczél's representation theorem, for each strictly increasing $\oplus$ there exists a strictly monotone surjective function (generator) $g : [a,b] \to [0,\infty]$ for $\oplus$ such that $x \oplus y = g^{-1}(g(x)+g(y))$ and $g(\mathbf{0}) = 0$. If $\mathbf{0} = a$, then g is increasing generator and $g(a) = 0$, $g(b) = \infty$, and g is an isomorphism between $([a,b],\oplus)$ and $([0,\infty],+)$. For $\mathbf{0} = b$, the situation is opposite. Then, pseudo-multiplication defined by $x \odot y = g^{-1}(g(x) \cdot g(y))$ is only one such that $([a,b], \oplus, \odot)$ is semiring (convention $\infty \cdot 0 = 0$ is used).

More about notions of measures with values in semiring $([a,b], \oplus, \odot)$ ($\oplus$-decomposable measures) and construction of corresponding pseudo-integral can be found in [3] and [6].

Let $([a,b], \oplus, \odot)$ be a semiring of type II with generator $g : [a,b] \to [0,\infty]$. As it is shown in [4], for $\lambda \in (0,\infty)$ function g^λ is generator for the semiring $([a,b], \oplus_\lambda, \odot_\lambda)$ with $x \oplus_\lambda y = (g^\lambda)^{-1}(g^\lambda(x) + g^\lambda(y))$ and $x \odot_\lambda y = (g^\lambda)^{-1}(g^\lambda(x) \cdot g^\lambda(y)) = x \odot y$. Hence $([a,b], \oplus_\lambda, \odot_\lambda) = ([a,b], \oplus_\lambda, \odot)$. The following three theorems proved in [4] show that: (1) the semiring of type I can be obtained as a limit of family of g^λ-generated semirings $([a,b], \oplus_\lambda, \odot_\lambda)$ (where $\odot_\lambda = \odot$), (2) decomposable measure based on idempotent pseudo-addition with a continuous density can be obtained as a limit of family of decomposable measures m_λ based on generated pseudo-additions, (3) the pseudo-integral based on semiring $([a,b], \sup, \odot)$ with $\odot$ generated by g and on sup-decomposable measure with a continuous density is a limit of family of g-integrals.

Theorem 2 *Let $g : [a,b] \to [0,\infty]$ be a strictly decreasing generator of the semiring $([a,b], \oplus, \odot)$ of the type* II *and g^λ the function g on the power $\lambda \in (0,\infty)$. Then g^λ is a generator of the semiring $([a,b], \oplus_\lambda, \odot)$ and for every $\varepsilon > 0$ and every $(x,y) \in [a,b]^2$ there exists λ_0 such that $|x \oplus_\lambda y - \inf(x,y)| < \varepsilon$ for all $\lambda \geq \lambda_0$. For g increasing, the same result holds for* sup.

Denote by $\mathcal{B}_{[0,\infty]}$ the σ-algebra of Borel subsets of the interval $[0,\infty]$, and denote by μ the usual Lebesgue measure on $\mathbb{R}$.

Remark 1 *If m is a $\oplus$-decomposable measure, where $\oplus$ is generated by g, then $\mu = g \circ m$ is a σ-additive measure, and $m = g^{-1} \circ \mu$ holds.*

Theorem 3 *Let m be a* sup*-decomposable measure on $([0,\infty], \mathcal{B}_{[0,\infty]})$, where $m(A) = \operatorname*{ess\,sup}_{\mu}\{\varphi(x) \mid x \in A\}$, where $\varphi : [0,\infty] \to [0,\infty]$ is a continuous density. Then for any generator g there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-decomposable measures on $([0,\infty], \mathcal{B}_{[0,\infty]})$, where $\oplus_\lambda$ is generated by g^λ, $\lambda \in (0,\infty)$, such that $\lim_{\lambda\to\infty} m_\lambda = m$ (i.e. $\lim_{\lambda\to\infty} m_\lambda(A) = m(A)$, for all $A \in \mathcal{B}_{[0,\infty]}$).*

Theorem 4 *Let $([0,\infty], \sup, \odot)$ be a semiring with $\odot$ generated by generator g. Let m be the same as in the theorem* 3*. Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-decomposable measures, where $\oplus_\lambda$ is generated by g^λ, $\lambda \in (0,\infty)$, such that for every continuous function $f : [0,\infty] \to [0,\infty]$*

$$\int^{\sup} f \odot dm = \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot dm_\lambda = \lim_{\lambda\to\infty} \left(g^{-1}\right)^\lambda \left(\int (g^\lambda \circ f) \odot dx \right).$$

For the sake of completeness, some properties of σ-$\oplus$-measures and appropriate integrals necessary for further consideration will be given (see [6], chapters 2 and 8). Let m be an σ-$\oplus$-measure on Ω with values in the semiring

$([0,\infty],\oplus,\odot)$, such that $m(\Omega)<\infty$. We adopt the convention $\infty\cdot 0=0$. Let $\chi_A:\Omega\to[0,\infty]$ denotes *pseudo-characteristic function* of a set $A\subset\Omega$, i.e. $\chi_A(x)=\begin{cases}\mathbf{0} & , \quad x\notin A\\ \mathbf{1} & , \quad x\in A\end{cases}.$

Theorem 5 *Let m be a σ-$\oplus$-decomposable measure. For any $c\in[0,\infty]$, any measurable $A\subseteq\Omega$, and any bounded measurable functions $f:\Omega\to[0,\infty)$ and $g:\Omega\to[0,\infty)$ is satisfied*

1. $m(A)=\int_\Omega^{\oplus}\chi_A(x)\odot dm,$

2. $\int_\Omega^{\oplus}(c\odot f)\odot dm=c\odot\int_\Omega^{\oplus}f\odot dm,$

3. $\int_\Omega^{\oplus}(f\oplus g)\odot dm=\int_\Omega^{\oplus}f\odot dm\oplus\int_\Omega^{\oplus}g\odot dm,$

4. $f\le g \quad\Rightarrow\quad \int_\Omega^{\oplus}f\odot dm\le\int_\Omega^{\oplus}g\odot dm.$

Theorem 6 *For the* sup*-measure m and a family of functions $f_j:\Omega\to[0,\infty)$, $j\in J$ holds*

$$\int_\Omega^{\sup}(\sup_{j\in J}f_j)\odot dm=\sup_{j\in J}\int_\Omega^{\sup}f_j\odot dm.$$

Let $\mathcal{F}$ be a collection of all closed subsets of $[0,\infty]$. Analogously as in theorem 1.7.7. in [7], for the sup-integral based on semiring $([0,\infty],\sup,\odot)$ with $\odot$ generated by continuous function g, the following theorem holds.

Theorem 7 *Let m be a $[0,\infty]$-valued, completely maxitive, $\mathcal{F}$-smooth* sup*-measure on $[0,\infty]$, i.e. $m(\emptyset)=0$, $m(\bigcup_{j\in J}A_j)=\sup_{j\in J}m(A_j)$ for every family A_j, $j\in J$ of measurable sets A_j, and $m(\bigcap_{n\in\mathbb{N}}F_n)=\inf_{n\in\mathbb{N}}m(F_n)$ for every decreasing sequence F_n, $n\in\mathbb{N}$ of elements of $\mathcal{F}$. Then, for every family of functions $f_j:[0,\infty]\to[0,\infty)$, $j\in J$ closed under formation of minimums is satisfied*

$$\int_\Omega^{\sup}(\inf_{j\in J}f_j)\odot dm=\inf_{j\in J}\int_\Omega^{\sup}f_j\odot dm,$$

where is $\odot$ generated by continuous function g.

4 Large deviation convergence of σ-$\oplus$-measures at rate $g - r_n$

Motivated by theorems 3 and 4, we shall now consider the convergence of σ-$\oplus_{r_n}$-decomposable measures m_n on $[0,\infty]$ with property $m_n([0,\infty]) = \mathbf{1}$ to the sup-decomposable measure m on $[0,\infty]$ with property $m([0,\infty]) = \mathbf{1}$, in the sense of large deviation principle (see [5], [3] and [6]).

Denote by $\mathcal{B}_{[0,\infty]}$ the Borel σ-algebra of subsets of $[0,\infty]$ (with usual topology on $[0,\infty]$). Let $\mathcal{O}$ and $\mathcal{F}$ denote families of open and closed sets in $[0,\infty]$ respectively.

Let $I = ([0,\infty], \oplus, \odot)$ be a semiring of type II with $\oplus$ and $\odot$ generated by continuous, strictly increasing function $g : [0,\infty] \to [0,\infty]$, so that $g(0) = 0$ and $g(\mathbf{1}) = 1$, where 0 and $\mathbf{1}$ are neutral elements for $\oplus$ and $\odot$ respectively.

Let $S = ([0,\infty], \sup, \odot)$ be a semiring with same operation $\odot$ as in semiring I (and 0 is also the neutral element for sup). Let $m : \mathcal{B}_{[0,\infty]} \to [0,\infty]$ be a completely maxitive, $\mathcal{F}$-smooth sup-measure on $[0,\infty]$ (see theorem 7) with property $m([0,\infty]) = \mathbf{1}$. Let r_n, $n \in \mathbb{N}$ be a sequence of real numbers greater than 1 satisfying $\lim\limits_{n\to\infty} r_n = \infty$. According the section 3, g^{r_n} is a generator of the semiring $([0,\infty], \oplus_{r_n}, \odot)$.

Let m_n, $n \in \mathbb{N}$ be a sequence of σ-$\oplus_{r_n}$-decomposable measures on the measure space $([0,\infty], \mathcal{B}_{[0,\infty]})$ with property $m_n([0,\infty]) = \mathbf{1}$, i.e.

$m_n(\emptyset) = 0$,

$m_n([0,\infty]) = \mathbf{1}$,

$m_n(\bigcup\limits_{i\in\mathbb{N}} A_i) = \bigoplus\limits_{i\in\mathbb{N}}{}_{r_n} m_n(A_i)$ for any collection of pairwise disjoint sets $A_i \in \mathcal{B}_{[0,\infty]}$, $i \in \mathbb{N}$.

Definition 3 *The sequence m_n, $n \in \mathbb{N}$* large deviation converge *at rate $g - r_n$ to m (*LD converge*, for short) if for all $f \in C_b^+(\mathbb{R}^+)$ is satisfied*

$$\lim_{n\to\infty} \int_{[0,\infty]}^{\oplus_{r_n}} f \odot dm_n = \int_{[0,\infty]}^{\sup} f \odot dm. \tag{2}$$

Large deviation convergence will be denoted as $m_n \underset{g-r_n}{\overset{ld}{\to}} m$, or $m_n \overset{ld}{\to} m$ shortly in the case when there is no confusion about the rate of convergence. The phrase "large deviation" will be shortly denoted as LD.

Remark 2 *If the limit value* (2) *exists, it is uniquely determined.*

We adopt the notation: $\quad \|f\|_{g-\oplus_{r_n}} = \int\limits_{[0,\infty]}^{\oplus_{r_n}} f \odot dm_n.$

Theorem 8 *If the sequence m_n, $n \in \mathbb{N}$ of $\oplus_{r_n}$-decomposable measures LD converge at rate $g - r_n$ to* sup*-decomposable measure m, then:*

(a) *for arbitrary open set* $O \subseteq [0, \infty]$ *holds*

$$\liminf_{n\to\infty} m_n(O) \geq m(O), \tag{3}$$

(b) *for arbitrary closed set* $F \subseteq [0, \infty]$ *holds*

$$\limsup_{n\to\infty} m_n(F) \leq m(F). \tag{4}$$

Considering theorem 1, some questions are imposed.

Problem 1 *Is the converse of theorem* 8 *true?*

Problem 2 *Is it true that inequalities* (3) *and* (4) *implies*

(A) *for arbitrary* $h \in \underline{C}_b^+([0, \infty])$ *holds*

$$\liminf_{n\to\infty} \|h\|_{g-r_n} \geq \int_{[0,\infty]}^{\sup} h \odot dm,$$

(B) *for arbitrary* $f \in \overline{C}_b^+([0, \infty])$ *holds*

$$\liminf_{n\to\infty} \|f\|_{g-r_n} \leq \int_{[0,\infty]}^{\sup} f \odot dm,$$

or under which additional conditions is it satisfied, and is the converse true?

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