LAW OF LARGE NUMBERS IN THE PSEUDO-PROBABILITY SPACES

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Abstract

We are using the conception and the theorems of the theory pseudo-probability.

We consider the sequence of independent variables in the pseudo-probability spaces and we give the law of large numbers on corresponding semirings $(I, \oplus, \odot)$.

AMS Mathematics Subject Classification (1991): 28E10, 60F05
Key words and phrases: pseudo - operation, $g-$ integral, pseudo-integral, pseudo-probability, quasi-arithmetic means.

1. The Preliminaries

Let is $(I, \oplus, \odot)$ a semiring.
Let Ω be a non-empty set. Let Σ be a $\sigma-$algebra of subsets of Ω.

In [3], the pseudo-integral of a bounded measurable function (for decomposable measure m) $f : \Omega \to I$ is defined. For the case II), the pseudo-integral reduces on $g-$integral, i.e.,

$$\int^{\oplus} f \odot dm = g^{-1}(\int\limits_R g(f(x))dx).$$

Pseudo - probability is a function $\mathbf{P} : \sum \to I$ with the properties $\mathbf{P}(\emptyset) = \mathbf{0}$, $\mathbf{P}(\Omega) = \mathbf{1}$ and

$$\mathbf{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} \mathbf{P}(A_i),$$

where $\{A_i\}_{i \in N}$ is a sequence of pairwise disjoint sets from $\sum$.
The triple $(\Omega, \Sigma, \mathbf{P})$ is *pseudo-probability space.*

In the case II), we have $\mathbf{P}(A) = g^{-1}(p(A))$, where p is usual probability.

The function $X : \Omega \to I$ is *pseudo-variable* if $\{\omega \in \Omega : X(\omega) \prec x\} \in \Sigma$ for all $x \in I$.

We also define the *distribution function* F of pseudo-variable X, as $F_X(x) = \mathbf{P}(X \prec x)$.

If there exists a function ϕ that holds $F_X(x) = \int\limits_{X^{-1}((.,x))}^{\oplus} \phi_X \odot d\mathbf{P}$, then we say that ϕ_X is *density function.*

The pseudo-expectation of the pseudo-variable X we define with

$$\mathbf{E}(X) = \int^{\oplus} x \odot \phi_X \odot d\mathbf{P}.$$

In the case II) is $\mathbf{E}(X) = g^{-1}\left(\int\limits_0^{\infty} g\left(x\right) \cdot g\left(\phi_X\left(x\right)\right) dx\right)$.

The pseudo-variable X and Y are *independent* if holds $\phi_{X,Y}(x,y) = \phi_X(x) \odot \phi_Y(y)$.

The sequence $\{X_n\}$ of pseudo-variables *converges in the pseudo-probability* $\mathbf{P}$, towards X, denoted $X_n \xrightarrow{\mathbf{P}} X$, if for all $\varepsilon > 0$ we have $\lim\limits_{n\to\infty} \mathbf{P}(\{\omega \in \Omega : d(X_n(\omega), X(\omega)) \geq \varepsilon\}) = \mathbf{0}$.

More details on convergences is given in [6].

2. The law of large numbers

Let g be the continuous strictly monotonic function. Then, we say for

$$S_n(x_1, x_2, ..., x_n) = g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right), \quad n \in N$$

that they are "quasi-arithmetic means".

In special cases, we have the table

$f(x)$	$S_n(x_1, x_2, ..., x_n)$	means
x	$\frac{1}{n}\sum\limits_{i=1}^{n} x_i$	arithmetic
x^2	$[\frac{1}{n}\sum\limits_{i=1}^{n} x_i^2]^{1/2}$	quadratic
x^α	$[\frac{1}{n}\sum\limits_{i=1}^{n} x_i^\alpha]^{1/\alpha}$	root-power
x^{-1}	$[\frac{1}{n}\sum\limits_{i=1}^{n} \frac{1}{x_i}]^{-1}$	harmonic
$\log x$	$[\prod\limits_{i=1}^{n} x_i]^{1/n}$	geometric
$e^{\alpha x}$	$\frac{1}{\alpha}\ln[\frac{1}{n}\sum\limits_{i=1}^{n} e^{\alpha x_i}]$	exponential

We consider the semiring II), in which the metric is defined with $d(x, y) = |g(x) - g(y)|$.

Theorem 1 *Let $X_1, X_2, ...$ be a sequence of independent pseudo-variables identically distributed,* $\mathbf{E}(X_n) = a, n = 1, 2, ...$ *. Then* $S_n \xrightarrow{\mathbf{P}} a$.

Proof. We prove that the following holds: $\lim\limits_{n\to\infty} \mathbf{P}(\{d(S_n, a) \geq \varepsilon\}) = \mathbf{0}$, for all $\varepsilon > 0$.

$$\begin{aligned}
& d(\mathbf{P}(\{d(S_n, a) \geq \varepsilon\}), \mathbf{0}) \\
= \; & |g(\mathbf{P}(\{d(S_n, a) \geq \varepsilon\})) - g(\mathbf{0})| \\
= \; & |p(\{d(S_n, a) \geq \varepsilon\})) - 0| \\
= \; & p(\{d(S_n, a) \geq \varepsilon\})) \\
= \; & p(\{|g(S_n) - g(a)| \geq \varepsilon\}) \\
= \; & p(\{|g(S_n) - g(a)| \geq \varepsilon\}) \\
= \; & p(\{|\frac{1}{n}\sum_{i=1}^{n} g(X_i) - g(a)| \geq \varepsilon\}).
\end{aligned}$$

As the variables $Y_i = g(X_i)$, $i = 1, ..., n$ satisfy the usual weak law of large numbers, this statement follows, i.e. $p(|\frac{1}{n}\sum\limits_{i=1}^{n} g(X_i) - g(a)| \geq \varepsilon\}) \to 0$.

□

References

[1] A. Chateauneuf, Decomposable measures, distorted probabilities and concave capacities, (to appear),

[2] J. L. Marichal, On an axiomatization of the quasi-arithmetic means values without the symmetry axiom, (to appear).

[3] E. Pap, *Null-Additive Set Functions*, Kluwer, Dordrecht; Ister Science, Bratislava, (1995).

[4] N. M. Ralević, The Pseudo-Probabilistic Spaces, (to appear).

[5] N. M. Ralević, Pseudo-analysis and applications on solution nonlinear equations, Ph. D. Thesis, PMF Novi Sad (1997).

[6] N. Ralević, The pseudo-probability, Zb. rad. Prim'98 (to appear).

[7] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, Elsevier-North Holland, New York, (1983).