The probability defined on semirings

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Abstract

A classical probability is the mapping from $\sigma-$algebra into the $(\mathbf{R}_0^+, +, \cdot)$. We replace $(\mathbf{R}_0^+, +, \cdot)$ by a semiring $(I, \oplus, \odot)$, and introduce the notion of pseudo-probability space. We state the corresponding convergence theorems, and give the law of large numbers on $(I, \oplus, \odot)$.

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1. Preliminaries

We briefly present some notions from the pseudo-analysis (see [3], [7]).

Let the order $\preceq$ be defined on a set $I \neq \emptyset$, and $\emptyset \neq I^* \subset I$.

Definition 1 *The* **pseudo-operation** *is a binary operation* $* : I \times I \to I$ *which is commutative, associative, positively nondecreasing ($x \preceq y$ implies $x * u \preceq y * u$, $u \in I^*$) and for which there exists a neutral element e.*

The element $u \in I$ is the **null element** of the operation $* : I^2 \to I$ if for any $x \in I$, $x * u = u * x = u$ holds.

Definition 2 *Let $\oplus$ and $\odot$ be two pseudo-operations defined on the ordered set $(I, \preceq)$, with $\mathbf{0}$ and $\mathbf{1}$ as neutral elements, respectively. Let $I^{\oplus} = I$, for the first operation, and $I^{\odot} = \{x \in I : \mathbf{0} \preceq x\}$, for the second operation. If $\odot$ is a distributive operation with respect to pseudo-operation $\oplus$ and $\mathbf{0}$ is a null element of the operation $\odot$, we say that the triplet $(I, \oplus, \odot)$ is a* **semiring***.*

The semiring $(I, \oplus, \odot)$ will be denoted by $I^{\oplus,\odot}$.

Let I be a subinterval of $[-\infty, +\infty]$ (we will take usually closed subintervals). Then we name the operations $\oplus$ and $\odot$ **pseudo-addition** and **pseudo-multiplication**.

Pseudo-operation $*$ is **idempotent** if for any $x \in I$, $x * x = x$. holds.

In this paper we will consider semirings with the following continuous operations:

Case I) a)

$$(i) \quad x \oplus y = \min(x, y), \quad x \odot y = x + y,$$

on the interval $(-\infty, +\infty]$. We have $\mathbf{0} = +\infty$ and $\mathbf{1} = 0$. The idempotent operation min induces a partial (full) order in the following way: $x \leq y$ if and only if $\min(x, y) = y$. Hence this order is opposite to the usual order on the interval $(-\infty, +\infty]$. We denote this semiring by $(-\infty, +\infty]^{\min,+}$.

$$(ii) \quad x \oplus y = \max(x, y), \quad x \odot y = x + y,$$

on the interval $[-\infty, +\infty)$. We have $\mathbf{0} = -\infty$ and $\mathbf{1} = 0$.

b)

$$(i) \quad x \oplus y = \min(x, y), \quad x \odot y = x \cdot y,$$

on the interval $(0, +\infty]$. We have $\mathbf{0} = \infty$ and $\mathbf{1} = 1$.

$$(ii) \quad x \oplus y = \max(x, y), \quad x \odot y = x \cdot y,$$

on the interval $[0, +\infty)$. We have $\mathbf{0} = 0$ and $\mathbf{1} = 1$.

Case II) Semirings with pseudo-operations defined by monotone and continuous generator g (see [3])

$$x \oplus y = g^{-1}(g(x) + g(y)), \quad x \odot y = g^{-1}(g(x)g(y)),$$

with the convention $0 \cdot (+\infty) = 0$, on the interval $[a, b]$. We have $\mathbf{0} = a$ or $\mathbf{0} = b$. This order is defined as $x \preceq y \Leftrightarrow g(x) \leq g(y)$.

Case III) a) Let $\oplus = \max$ and $\odot = \min$ on the interval $[-\infty, +\infty]$. We have $\mathbf{0} = -\infty$ and $\mathbf{1} = +\infty$.

b) Let $\oplus = \min$ and $\odot = \max$ on the interval $[-\infty, +\infty]$. We have $\mathbf{0} = +\infty$ and $\mathbf{1} = -\infty$.

For the case I) a) (i) on the interval $(-\infty, +\infty]$ we introduce a metric

$$d(x, y) = |e^{-x} - e^{-y}|. \tag{1}$$

For the case II) on the interval $[a, b]$ we introduce a metric

$$d(x, y) = |g(x) - g(y)|. \tag{2}$$

For the case III) b) on the interval $[-\infty, +\infty]$ we introduce a metric

$$d(x, y) = \frac{2}{\pi} |\operatorname{arctg} x - \operatorname{arctg} y|. \tag{3}$$

2. The pseudo-probability

Let $(I, \oplus, \odot)$ be a semiring.

Let Ω be a non-empty set. Let Σ be a $\sigma-$algebra of subsets of Ω.

In [3], the pseudo-integral of a bounded measurable function (for decomposable measure m) $f : \Omega \to I$ is defined. For the case II), the pseudo-integral reduces to $g-$integral, i.e.,

$$\int_{\Omega}^{\oplus} f \odot dm = g^{-1}(\int_{\Omega} g(f(x))dx).$$

Definition 3 *Let* $\sum$ *be* σ *- algebra of subsets of a set* Ω. **Pseudo - probability** *is a function* $\mathbf{P} : \sum \to I$ *with the properties*

(a) $\mathbf{P}(\emptyset) = \mathbf{0}$ *and* $\mathbf{P}(\Omega) = \mathbf{1}$,

(b) $\mathbf{P}(A \cup B) = \mathbf{P}(A) \oplus \mathbf{P}(B)\,, \quad A, B \in \sum, \quad A \cap B = \emptyset,$

(c) $A_i \in \sum,\ i \in N, \quad A_i \subseteq A_{i+1},\ i \in N \quad \Rightarrow \quad \lim_{i\to\infty} \mathbf{P}(A_i) = \mathbf{P}\left(\bigcup_{i=1}^{\infty} A_i\right).$

The triple $(\Omega, \Sigma, \mathbf{P})$ *is a* **pseudo-probability space***.*

An equivalent definition of pseudo - probability is obtained if the conditions (b) and (c) are replaced by the condition

$$\mathbf{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} \mathbf{P}(A_i)\,,$$

where $\{A_i\}_{i \in N}$ is a sequence of pairwise disjoint sets from $\sum$ ($\sigma - \oplus -$ additivity of function $\mathbf{P}$).

Let us notice that the pseudo-probability is a special case of decomposable measure.

In the case II), we have $\mathbf{P}(A) = g^{-1}(p(A))$, where p is the usual probability. Then we say that $\mathbf{P}$ is the distorted probability (see [1]).

The function $X : \Omega \to I$ is a **pseudo-variable** if $X^{-1}((\cdot, x)) = \{\omega \in \Omega : \ X(\omega) \prec x\} = \{X \prec x\} \in \Sigma$ for all $x \in I$.

We also define the **distribution function** F of pseudo-variable X, as $F_X(x) = \mathbf{P}(\{X \prec x\})$.

Let $\sigma(I)$ be a minimal $\sigma-$algebra containing open balls in separable metric space (I, d). Let m be a decomposable measure defined in the measurable space $(I, \sigma(I))$.

If there exists a function ϕ that holds $F_X(x) = \int_I^{\oplus} \phi_X dm$ then we say that ϕ is the density function.

Definition 4 *The sequence* $\{X_n\}$ *of pseudo-variables* **converges in the pseudo-probability P**, *towards* X, *denoted* $X_n \xrightarrow{\mathbf{P}} X$, *if for all* $\varepsilon > 0$ *we have*

$$\mathbf{P}(\{\omega \in \Omega : d(X_n(\omega), X(\omega)) \geq \varepsilon\}) \to \mathbf{0}.$$

Definition 5 *The sequence* $\{X_n\}$ *of pseudo-variables* **converges almost surely** *towards* X, *denoted* $X_n \xrightarrow{a.s.} X$ *if we have*

$$\mathbf{P}(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}) = \mathbf{1},$$

i.e.

$$\mathbf{P}(\{\omega \in \Omega : X_n(\omega) \nrightarrow X(\omega)\}) = \mathbf{0}.$$

In the idempotent cases I) and III), we have (see [6]):

Theorem 1 *Let* X_n *and* X *denote pseudo variables. Then* $X_n \xrightarrow{\mathbf{P}} X$ *implies* $X_n \xrightarrow{a.s.} X$.

In the not idempotent case II), we have

Theorem 2 *Let* X_n *and* X *denote pseudo variables. Then* $X_n \xrightarrow{a.s.} X$ *implies* $X_n \xrightarrow{\mathbf{P}} X$.

Proof. From $\mathbf{P}(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}) = \mathbf{1}$, i.e. $g^{-1}(p(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\})) = \mathbf{1}$, we obtain $p(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}) = 1$, because $g(\mathbf{1}) = 1$. In metric space (I, d), $X_n(\omega) \to X(\omega)$ is equivalent with $(\forall \delta > 0)(\exists n_0 \in N)(\forall n \in N) n \geq n_0 \Rightarrow \delta > d(X_n(\omega), X(\omega)) = |g(X_n(\omega)) - g(X(\omega))|$, i.e. the sequence $\{g(X_n)\}$ of random variables converges almost surely towards $g(X)$. In the usual probability theory, almost surely the convergence implies the convergence in the probability, so we have that for all $\varepsilon > 0$ holds:

$$p(\{\omega \in \Omega : |g(X_n(\omega)) - g(X(\omega))| \geq \varepsilon\}) \to 0, \ n \to \infty.$$

Finally, from
$\mathbf{P}(\{\omega \in \Omega : d(X_n(\omega), X(\omega)) \geq \varepsilon\}) \to \mathbf{0}$
$\Leftrightarrow 0 = \lim_{n\to\infty} d(\mathbf{P}(\{\omega \in \Omega : d(X_n(\omega), X(\omega)) \geq \varepsilon\}), \mathbf{0})$
$= \lim_{n\to\infty} |g(\mathbf{P}(\{\omega \in \Omega : d(X_n(\omega), X(\omega)) \geq \varepsilon\})) - g(\mathbf{0})|$
$= \lim_{n\to\infty} p(\{\omega \in \Omega : d(X_n(\omega), X(\omega)) \geq \varepsilon\})$
$= \lim_{n\to\infty} p(\{\omega \in \Omega : |g(X_n(\omega)) - g(X(\omega))| \geq \varepsilon\})$, we obtain that the sequence $\{X_n\}$ of pseudo-variables converges in the pseudo-probability **P** towards X. □

The **pseudo-expectation** of the pseudo-variable X we define with

$$\mathbf{E}(X) = \int_I^{\oplus} x \odot \phi_X(x) \odot dm.$$

In the case II) we have $\mathbf{E}(X) = g^{-1}\left(\int_I g(x) \cdot g(\phi_X(x))\, dx\right)$, where $dx = d(g \circ m)$ is the Lebesgue measure.

The pseudo-variable X and Y are **independent** if it holds $\phi_{X,Y}(x,y) = \phi_X(x) \odot \phi_Y(y)$.

3. The law of large numbers

Let g be the continuous strictly monotonic function. Then, we say for

$$S_n(x_1, x_2, ..., x_n) = g^{-1}(\frac{1}{n}\sum_{i=1}^{n} g(x_i)), \quad n \in N$$

that they are the "quasi-arithmetic means".

In the special cases, we have the table

$f(x)$	$S_n(x_1, x_2, ..., x_n)$	means
x	$\frac{1}{n}\sum_{i=1}^{n} x_i$	arithmetic
x^2	$[\frac{1}{n}\sum_{i=1}^{n} x_i^2]^{1/2}$	quadratic
x^α	$[\frac{1}{n}\sum_{i=1}^{n} x_i^\alpha]^{1/\alpha}$	root-power
x^{-1}	$[\frac{1}{n}\sum_{i=1}^{n} \frac{1}{x_i}]^{-1}$	harmonic
$\log x$	$[\prod_{i=1}^{n} x_i]^{1/n}$	geometric
$e^{\alpha x}$	$\frac{1}{\alpha}\ln[\frac{1}{n}\sum_{i=1}^{n} e^{\alpha x_i}]$	exponential

The following theorem holds (see [8]):

Theorem 3 *Let $X_1, X_2, ...$ be a sequence of independent pseudo-variables identically distributed, $\mathbf{E}(X_n) = a, n = 1, 2, ...$. Then $S_n \xrightarrow{\mathbf{P}} a$.*

References

[1] A. Chateauneuf, Decomposable measures, distorted probabilities and concave capacities, (to appear),

[2] J. L. Marichal, On an axiomatization of the quasi-arithmetic means values without the symmetry axiom, (to appear).

[3] E. Pap, *Null-Additive Set Functions*, Kluwer, Dordrecht; Ister Science, Bratislava, (1995).

[4] E. Pap, N. Ralević, Pseudo operations on finite intervals, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. (too appear).

[5] N. M. Ralević, The Pseudo-Probabilistic Spaces, (too appear).

[6] N. M. Ralević, The pseudo-probability, Zb. rad. Prim'98 (to appear).

[7] N. M. Ralević, Pseudo-analysis and applications on solution nonlinear equations, Ph. D. Thesis, PMF Novi Sad (1997).

[8] N. M. Ralević, T. Grbić, Lj. M. Nedović, Law of large numbers in the pseudo-probability spaces, Zb. rad. Prim'98 (to appear).

[9] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, Elsevier-North Holland, New York, (1983).