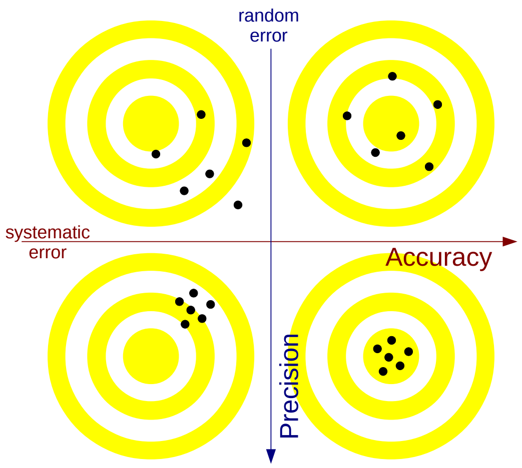
Sub-Pixel Precision Measurement in the Point-Sampling Model

Topics

- What is important for a measure?
  - accuracy (bias), precision, sampling-invariance
- How does filtering affect the measurement?
- The point-sampling model:
  - image formation, band limit, sampling, Fourier analysis
- Soft clipping
- Measurement:
  - area
  - perimeter
  - curvature
  - bending energy
  - Euler number (object count)

Accuracy vs. precision



precision is useless if measurement is biased

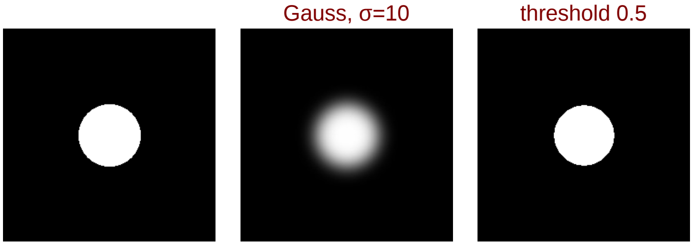
bias = lack of accuracy

Sampling invariance

- The choice of sampling grid should not influence the measurement result:
  - translation invariance
  - rotation invariance
  - scaling invariance
    - when counting pixels, a denser grid gives higher precision
- Methods we talk about today are invariant to scaling!
  - given a properly sampled band-limited function
  - thus: there is a minimum sampling density
  - band limit gives maximum attainable precision

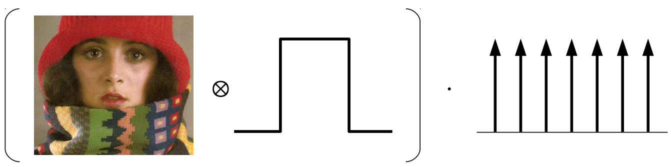
Effects of filtering

- Low-pass filtering always moves the edges inwards
  - (Inwards = in the direction of curvature)
  - This is also true for median filtering, for example!
- Edge-preserving smoothing filters sometimes also move edges



The point-sampling model

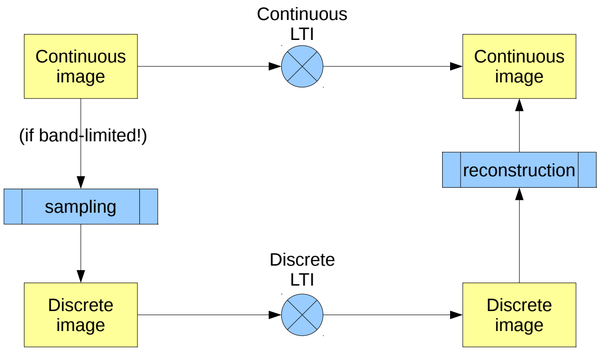
- Point sampling is what is assumed in signal theory
- Point sampling is only useful if the image is band limited
  - otherwise we get aliasing
  - *sampling frequency* > 2 · *band limit* (Nyquist)
- CCDs do not point-sample
  - but: can be modelled by a uniform filter followed by point sampling



Band-limited images

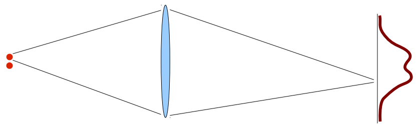
- Any optical image formation system imposes a band limit
- A sampled band-limited image *exactly* represents the continuous band-limited image
  - if sampled properly, of course
- The continuous band-limited image is a version of real world that lacks very high frequencies
  - i.e. small details
- This smooth image preserves large-scale geometric properties of the imaged objects, but not small scale ones
  - as in “the infinite coastline of Britain”

The sampling property

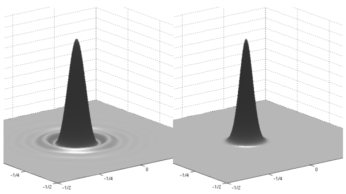


Optical image formation

- Image formation system (e.g. lenses) creates a band-limited image — imposes resolution

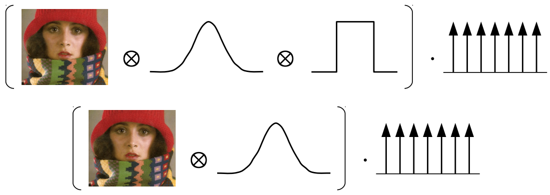


- Standard optics’ point-spread function (PSF) can be approximated by a Gaussian
  - ideal lens has Airy function for PSF
  - but lens imperfections are unavoidable



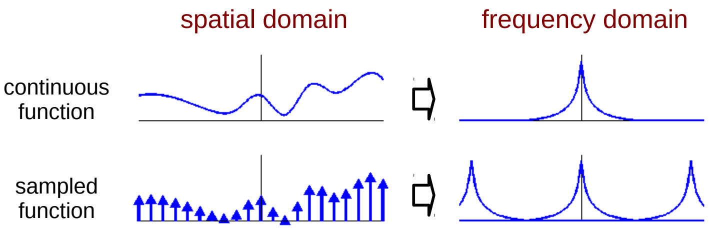
Optical image formation

- The image is smoothed by a PSF (convolution!) before sampling



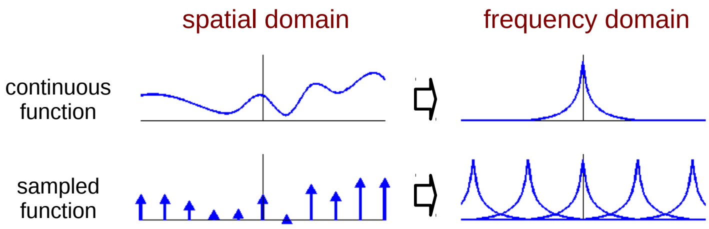
- Neither the smoothing nor the sampling change the total amount of light in the image

What happens in the Fourier domain



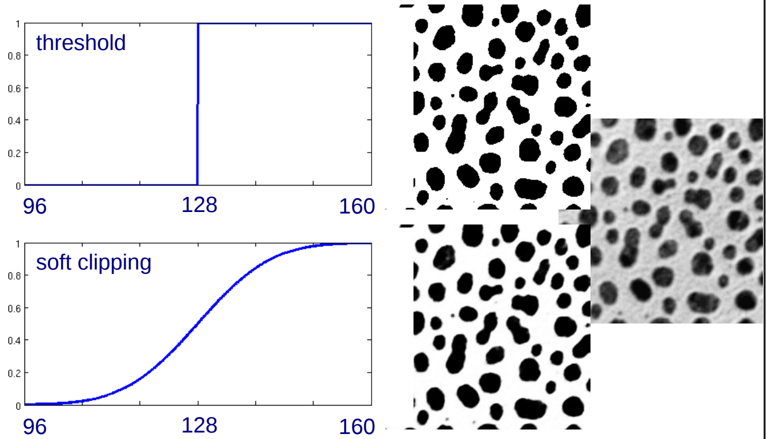
- The 0^{th} frequency is proportional to the total amount of light
- 0^{th} frequency is unaltered by sampling
- Sum of samples is equal (proportional) to integral over continuous function

What happens in the Fourier domain

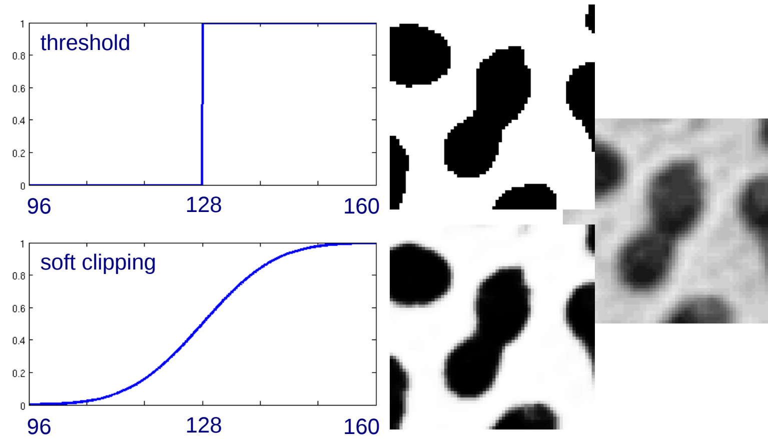


- But: aliasing can affect the 0^{th} frequency!
- Sum of samples is equal (proportional) to integral over continuous, band-limited function if sampled correctly

Threshold vs. soft clipping

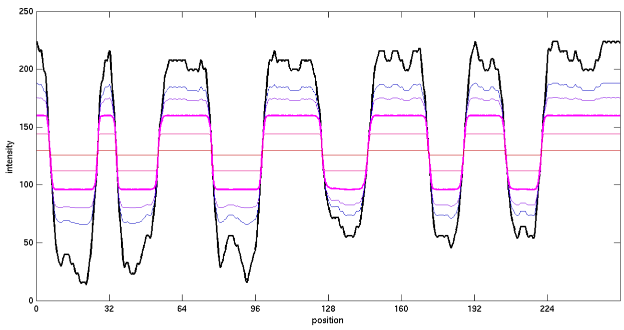


Threshold vs. soft clipping



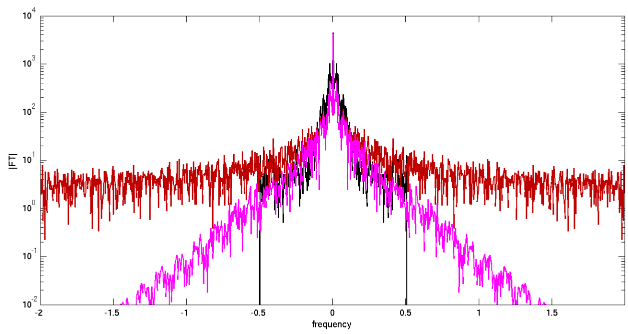
Soft clipping

- Selecting a proper range is important
  - too small: introduction of aliasing
  - too large: background and foreground not uniform



Soft clipping

- Interpolated 4x by padding the Fourier transform *before* soft clipping
  - input, soft clipping, threshold

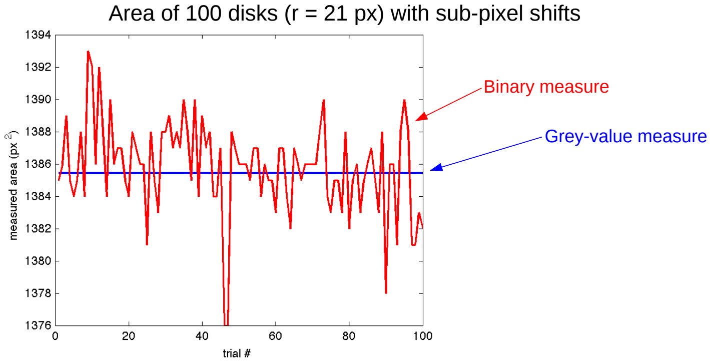


Possible measures

- Area (2D) / volume (3D)
  - integral over image (sum of grey values)
  - effectively dimensionality-independent
- Perimeter (2D) / surface area (3D)
  - we convert the problem to a volume problem
  - effectively dimensionality-independent
- (Isophote) curvature (2D/3D)
  - based on 2nd derivative along the contour
- Bending energy (2D/3D)
  - integrating squared curvature along contour
- Euler number (object count, 2D)
  - integral of curvature along contour is constant

2D area (ideal case)

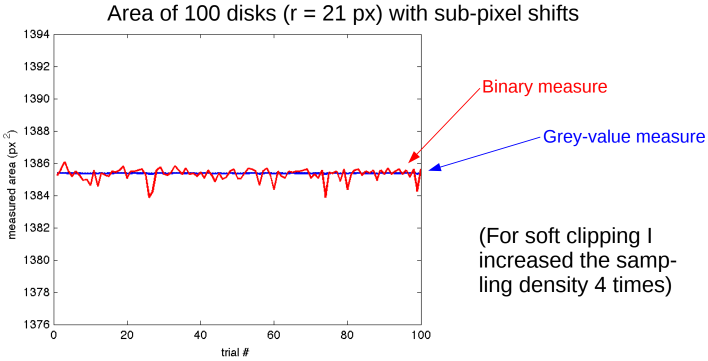
Area of 100 disks (r = 21 px) with sub-pixel shifts



Expected measure:	1385.442360 px^2	
Grey-value measure:	1385.442352 ± 0.000001 px^2	(std = 0.000006)
Binary measure:	1385.8 ± 0.6 px^2	(std = 2.9)

2D area (with soft clipping)

Area of 100 disks (r = 21 px) with sub-pixel shifts



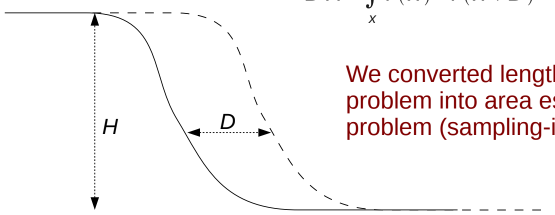
(For soft clipping I increased the sampling density 4 times)

Expected measure:	1385.442360 px²	
Grey-value measure:	1385.370 ± 0.003 px²	(std = 0.013)
Binary measure:	1385.31 ± 0.08 px²	(std = 0.39)

2D perimeter

- Given a grey-value object with a constant intensity *H*
- If we extend the object by a fixed distance *D*, the volume of the extension is given by: *P D H* (*P* = perimeter)

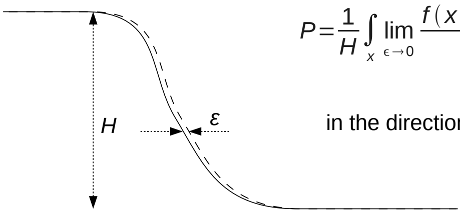
$$DH = \int_x f(x) - f(x+D)$$



We converted length estimation problem into area estimation problem (sampling-invariant!)

2D perimeter

- Given a grey-value object with a constant intensity *H*
- If we extend the object by a fixed distance ε, the area of the extension is given by: *P ε H* (*P* = perimeter)

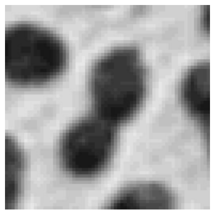


$$P = \frac{1}{H} \int_x \lim_{\epsilon \to 0} \frac{f(x) - f(x+\epsilon)}{\epsilon} dx$$

in the direction of the gradient....

2D perimeter

- Soft clipping
- Gradient magnitude $\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$
- Integration (sum)

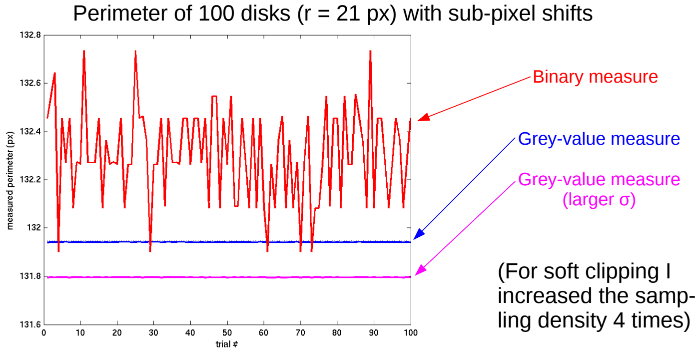


We're using Gaussian gradients!

$$\frac{\partial}{\partial x}(f \otimes G) = f \otimes \frac{\partial}{\partial x} G$$

2D perimeter

Perimeter of 100 disks (r = 21 px) with sub-pixel shifts



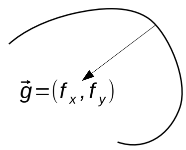
(For soft clipping I increased the sampling density 4 times)

Expected measure:	131.946891 px	
Grey-value measure:	131.9415 ± 0.0001 px	(std = 0.0006)
with larger σ:	131.7958 ± 0.0001 px	(std = 0.0006)
Binary measure:	132.04 ± 0.01 px	(std = 0.07)

Isophote curvature

- Contour direction: $\vec{c} = (-f_y, f_x)$

$$\theta = \arccos\left(\frac{-f_y}{|g|}\right) = \arcsin\left(\frac{f_x}{|g|}\right) = \arctan\left(\frac{-f_y}{f_x}\right)$$



- To differentiate along the curve:

$$\frac{d}{ds} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} = \frac{-f_y}{|g|}\frac{\partial}{\partial x} + \frac{f_x}{|g|}\frac{\partial}{\partial y}$$

- Curvature *κ* = derivative of *θ* along the curve

$$\kappa = \frac{d\theta}{ds} = -\frac{f_{xx} f_y^2 - 2 f_x f_y f_{xy} + f_{yy} f_x^2}{\left(f_x^2 + f_y^2\right)^{3/2}} = \frac{-f_{cc}}{|g|}$$

- 3D version more involved: eigenvalues of Hessian...

Bending energy

- Often used as a shape descriptor
- Given by integral along the perimeter of the square of curvature
- Integrate along perimeter by multiplying by |g| and integrating over the image

$$\text{B.E.} = \int_{\text{contour}} \kappa^2 \, d s = \iint_{\text{image}} \kappa^2 |g| \, d x \, d y = \iint_{\text{image}} \frac{f_{cc}^2}{|g|} \, d x \, d y$$

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Bending energy

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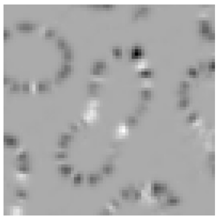
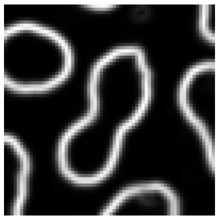
Erf-clip



f_{cc}



$|g|$

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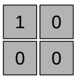
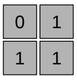
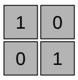
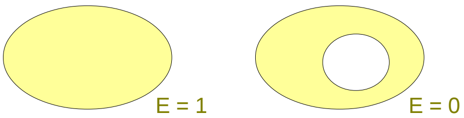
Euler number

- Euler number
  - In 2D: # of objects - # of holes
  - In 3D: # of objects - # of tunnels + # of cavities
- Gray's algorithm
  - Computes Euler number based on 2x2 image regions
  - $E = (C_1 – C_2 – 2C_3) / 4$
  - C1 = # of 2x2 regions with only 1 pixel set
  - C2 = # of 2x2 regions with 3 pixels set
  - C3 = # of 2x2 regions 2 pixels set in a diagonal









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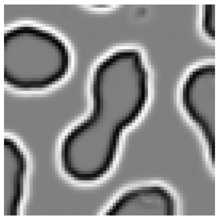
Euler number

- Integral of curvature along a closed contour is always 2π, a hole in an object contributes with -2π

$$\text{Euler number} = \frac{1}{2\pi} \int_{\text{contour}} \kappa \, d s = \frac{1}{2\pi} \iint_{\text{image}} \kappa |g| \, d x \, d y = \frac{1}{2\pi} \iint_{\text{image}} -f_{cc} \, d x \, d y$$

- Integral of second derivative in gradient direction also yields a constant 2π for a closed contour

$$\text{Euler number} = \frac{1}{2\pi} \iint_{\text{image}} f_{gg} \, d x \, d y$$



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Summary

- It is important to use unbiased measures
- Filtering can introduce bias
- Area/volume = integral over image
- Perimeter/surface area
  - obtained by converting to area measurement problem
- Curvature
  - computed through 2nd derivative along countour
  - bending energy & Euler number
- Prepare image by soft clipping
  - (equivalent to thresholding, but without loss of band limitation)