Image analysis with subpixel precision - The Coverage model

Part 3 - Feature extraction

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Feature estimation - some general observations

- Our aim is to obtain information about continuous real objects, having available their discrete - **coverage** - representation.
- Different numerical descriptors, such as area, perimeter, moments, of the objects are often of interest, for the tasks of shape analysis, classification, etc.
- Estimators should be adjusted/designed so that they utilize in a best way information preserved in a coverage representation.

Precision of features extracted from the coverage representation

1. Geometric moments
   - N. Sladoje and J. Lindblad. Estimation of Moments of Digitized Objects with Fuzzy Borders. ICIAP'05, LNCS-3617, pp. 188-195, Cagliari, Italy, 2005.
2. Perimeter
   - N. Sladoje and J. Lindblad. High Precision Boundary Length Estimation by Utilizing Gray-Level Information. IEEE Trans. on PAMI, Vol. 31, No. 2, pp. 357–363, 2009.
3. Signature of a shape
   - J. Chanussot, I. Nyström, N. Sladoje. Shape Signatures of Fuzzy Sets Based on Distance from the Centroid, PRL, 26(6), pp. 735-746, 2005.
4. Projection (diameter, elongation)
   - S. Dražić, J. Lindblad, N. Sladoje. Precise Estimation of the Projection of a Shape from a Pixel Coverage Representation. Proc. of ISPA 2011 (IEEE), pp. 569-574, Dubrovnik, Croatia, 2011.
5. Distances between sets (shape matching, image registration)
   - V. Ćurić, J. Lindblad, and N. Sladoje. Distance measures between digital fuzzy objects and their applicability in image processing. IWCIA2011, LNCS-6636, pp. 385-395, Madrid, Spain, 2011.

Feature estimation - some general observations

- Evaluation of an estimator should, in an ideal case, provide some relevant error bounds.
- Several estimators proposed for general fuzzy membership functions are (only) statistically evaluated (for some selection of cases) and improvements in terms of precision are observed.
- Generality is often an obstacle for derivation of stronger theoretical statements about the derived estimation methods.
- Membership function of a coverage model is restricted enough to allow derivation of error bounds for feature estimators.

Feature extraction - some general observations

Aggregation over α-cuts – a standard approach for fuzzy sets

Given a function $f : \mathcal{P}(X) \to \mathbb{R}$, which assigns a real valued "feature" to a **crisp subset** of an integer grid,

we can extends this function to $f : \mathcal{F}(X) \to \mathbb{R}$, so that it assigns a real valued feature to a **fuzzy subset** of an integer grid, using the equation

$$f(S) = \int_0^1 f(S_\alpha)d\alpha,$$

where S_α is an **α-cut** of a fuzzy set S, i.e., a **crisp set** that contains all the elements in X that have membership value in S greater than or equal to α:

$$F_\alpha = \{x \in X \mid \mu_F(x) \geq \alpha\}.$$

α-cutting is thresholding of the membership function at a level α.

Geometric moments - computation

Definition

The two-dimensional Cartesian moment, $m_{p,q}$ of a function $f(x,y)$ is defined as

$$m_{p,q} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)x^p y^q \, dxdy \ ,$$

for integers $p,q \geq 0$. The moment $m_{p,q}(S)$ has the order $p+q$.

Definition

Geometric moment $m_{p,q}$ of a digital image $f(i,j)$ is

$$m_{p,q} = \sum_i \sum_j f(i,j) i^p j^q \ ,$$

where (i,j) are points in the (integer) sampling grid.

Geometric moments of objects provide information about area, (hyper-)volume, centroid, principal axes, and a number of other features of the shape.

Geometric moments - error estimation

Crisp representation:

Theorem

The moments of a closed bounded set S, digitized in a grid with resolution r (the number of grid points per unit), can be estimated by

$$m_{p_1,p_2}(S) = \frac{1}{r^{p_1+p_2+2}}\,\tilde{m}_{p_1,p_2}(rS) + \mathcal{O}\left(\frac{1}{r}\right)$$

for $p_1 + p_2 \le 2$

Here rS denote a scaling of the continuous set S about the origin by the factor r. Scaling of the object can be used instead of changing resolution of a grid.

Geometric moments are multi-grid convergent.

Geometric moments - error estimation

Coverage representation:

Theorem

The moments of a closed and bounded 2D shape S can, for $p_1 + p_2 \le 2$, be estimated by

$$m_{p_1,p_2}(S) = \frac{1}{r_s^{p_1+p_2+2}}\,\tilde{\mathcal{M}}^{r_f}_{p_1,p_2}(r_s S) + \mathcal{O}\left(\frac{1}{r_s^2}\right) + \mathcal{O}\left(\frac{1}{r_s r_f}\right)$$

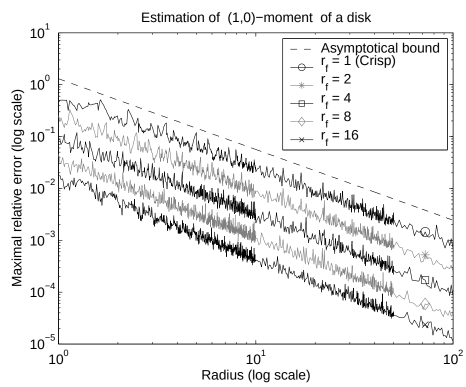
where $\tilde{\mathcal{M}}^{r_f}_{p_1,p_2}$ is (p_1,p_2)-geometric moment of $r_s S$ computed from its r_f-sampled coverage segmentation.

Geometric moments - error estimation

Observations:

- Once the spatial resolution is high enough to fully "exploit" the coverage values of pixels, using r_f^2 coverage values provides the same accuracy of moment estimation as increasing the (crisp) spatial resolution of the image r_f times.
- Even though r_f-sampled coverage representation is observed here, the results hold for ℓ-level quantized coverage digitization (with $\ell = r_f^2$).
- Extensions to nD and moments of higher orders are studied as well. Corresponding error bounds are derived (Course material).

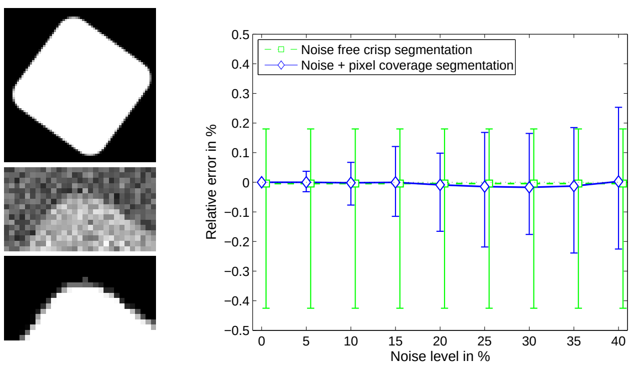
Geometric moments - error estimation



Plot of maximal observed error for first order moment estimation for different spatial and coverage resolutions, for a disk.

Area estimation error in a noisy environment

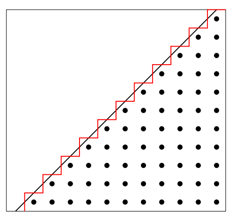
in combination with coverage segmentation



Left: (top) Synthetic test objects. (middle) Part of object with 30% noise added. (bottom) Coverage segmentation result for 30% noise. **Right:** Estimation errors for increasing levels of noise. Green is noise free crisp reference. Lines show averages for 50 observations and bars indicate max and min errors.

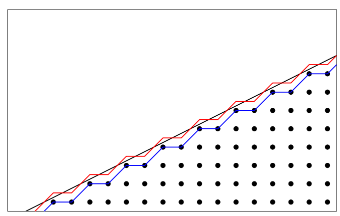
How to assign local step lengths

Using 4 edge directions.

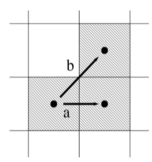


Digital edge $\sqrt{2}$ times longer than true edge.

Using 8 edge directions.



Edge 1.08 times longer than true edge.



Freeman 1970:
$a = 1, \quad b = \sqrt{2}$

Using $a = 1$, $b = \sqrt{2}$ lead to an **overestimate**.

Error minimization

- Observe straight lines of all directions.
- Restrict to lines with slopes $k \in [0, 1]$. Other cases follow by symmetries.
- Decide what error to minimize
  - The mean square error (MSE) minimization leads to estimators that, in average, perform well for lines of all directions.
  - The maximal error minimization leads to estimator with a better "controllable" error.
- Compute **optimal step lengths** to minimize the chosen error measure when estimating the length of straight segments of arbitrary direction.

- To minimize MSE: $a = 0.9481$ and $b = 1.3408$. Root Mean Square (RMS) Error is 2.33%.
- To minimize MaxErr: $a = 0.9604$ and $b = 1.3583$. Maximal Error is 3.95%.
- **The error does not decrease with increasing resolution**

Local and non-local estimators

Local estimators

Use **information from a small region** of the image to compute a local feature estimate. The global feature is computed by a summation of the local feature estimates over the whole image.

- Easy to implement
- Trivial to parallelize
- If a local change in the image, only that part has to be traversed to update the estimate
- Stable, if the local estimate is bounded

Non-local estimators

Use information from larger (unbounded) regions of the image.

- Difficult to parallelize, if at all possible
- Often of higher complexity (may be NP-hard)
- May suffer from stability problems
- Small change of the image requires global recomputation

Local and non-local estimators

- Local estimators have sufficiently many advantages compared to global ones, to deserve to be studied further.
- Local perimeter estimators are, however, **not** multigrid convergent, not even for straight edges.

The straight edge of a halfplane

Discrete, grey-scale, non-quantized

Observe a halfplane $H = \{(x, y) \mid y(x) \leq kx + m,\ k, m \in [0, 1]\}$, over an interval $x \in [0, N]$, $N \in \mathbb{Z}^+$.
Let I be the non-quantized pixel coverage digitization $I = \mathcal{D}(H)$ ($\Delta x = \Delta y = h = 1$ by definition.)

Then it holds that

$$y(i) = \sum_{j \geq 0} I(i, j) - 0.5$$

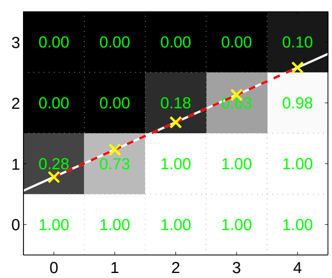
$$k(i) = y(i+1) - y(i) = k$$

$$l = \sqrt{N^2 + (kN)^2} = \sum_{i=0}^{N-1} \sqrt{1 + k(i)^2}$$

The length of the edge segment l is "estimated" with **no error**.

Non-quantized example

$l = 4.39$



s_c:	1.28		1.73		2.18		2.63		3.08
d_c:		0.45		0.45		0.45		0.45	
l_c:		1.10		1.10		1.10		1.10	

Example illustrating edge length estimation based on the difference d_c of column sums s_c for a segment ($N = 4$) of a halfplane edge given by $y \leq 0.45x + 0.78$.

$$s_c = \sum_{j \geq 0} I(c, j)\ ,\ d_c = s_{c+1} - s_c\ ,\ l_c = \sqrt{1 + d_c^2}$$

The straight edge of a halfplane

Discrete, grey-scale, quantized

Observe a halfplane $H = \{(x, y) \mid y(x) \leq kx + m,\ k, m \in [0, 1]\}$, over an interval $x \in [0, N]$, $N \in \mathbb{Z}^+$.
Let I be the **quantized** pixel coverage digitization $I = \mathcal{D}^n(H)$

Then

$$\tilde{l} = \sum_{c=0}^{N-1} \sqrt{1 + d_c^2}$$

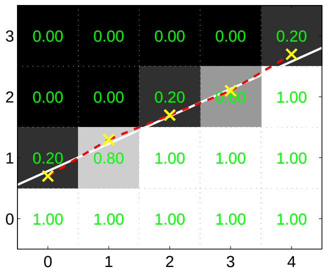
provides an estimate of the edge length l.

However, this is in general an overestimate (zig-zag steps). Scaling the estimate with an optimally chosen factor $\gamma_n < 1$, gives an estimate with a minimal error.

$$\hat{l} = \sum_{c=0}^{N-1} \gamma_n \sqrt{1 + d_c^2}$$

Quantized example

$\hat{l} = \gamma_5 * 4.49$



$\tilde{s}_c$:	1.20		1.80		2.20		2.60		3.20
$\tilde{d}_c$:		0.60		0.40		0.40		0.60	
$\tilde{l}_c$:		1.17		1.08		1.08		1.17	

Example illustrating edge length estimation based on the difference d_c of column sums s_c for a segment ($N = 4$) of a halfplane edge given by $y \leq 0.45x + 0.78$.

$$s_c = \sum_{j \geq 0} I(c,j)\ ,\ \ d_c = s_{c+1} - s_c\ ,\ \ l_c = \sqrt{1 + d_c^2}$$

Minimization of the maximal relative error

An edge is a linear combination of local steps

The edge segment $\boldsymbol{l} = (N, kN)$ can be expressed as a linear combination of two of the vectors, $\boldsymbol{S_i} = (1, \frac{i}{n})$, $\boldsymbol{S_j} = (1, \frac{j}{n})$, $i, j \in \{0, 1, \ldots, n\}$, having slopes $k_i = \frac{i}{n}$, $k_j = \frac{j}{n}$ such that $k_i \leq k \leq k_j$.
Its length on the interval $[0, N]$ can be estimated by

$$\hat{l} = \gamma_n \left(\frac{(j - nk)N}{j - i} S_i + \frac{(nk - i)N}{j - i} S_j \right)\ ,\ \text{where } S_i = \sqrt{1 + (\tfrac{i}{n})^2}\ .$$

Relative error of the length estimation

The relative error of the length estimation of the line segment with slope k, such that $k \in [\frac{i}{n}, \frac{j}{n}]$:

$$\varepsilon_{i,j}(k) = \frac{\hat{l} - l}{l} = \gamma_n \frac{(j - nk)S_i + (nk - i)S_j}{(j - i)\sqrt{1 + k^2}} - 1\ .$$

Minimization of the maximal relative error

Result [Sladoje and Lindblad, PAMI 2009]

The maximal error is minimized for

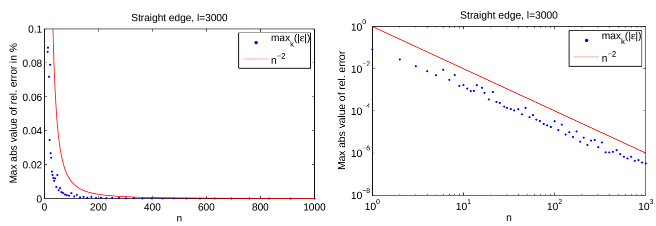
$$\gamma_n^q = \frac{2q}{q + \sqrt{(\sqrt{n^2 + q^2} - n)^2 + q^2}}\ ,\ \text{where } q = j - i\ .$$

The maximal error is $|\varepsilon| = 1 - \gamma_n^q$.
Quantization leads to $q > 1$. In 2D it holds that $q \leq 3$.

Asymptotic behaviour

Observing the estimation error as a function of the number of grey-levels n, we conclude that

$$|\varepsilon_n| = \mathcal{O}\left(\frac{1}{n^2}\right)\ .$$



(a) lin-lin scale (b) log-log scale

Asymptotic behaviour of the maximal error for straight edge length estimation using $\gamma_n = \gamma_n^1$; theoretical (line) and empirical (points) results.

Algorithm

Input: Pixel coverage values $\tilde{p}_i$, $i = 1, \ldots, 9$, from a 3×3 neighbourhood $T_{(c,r)}$.
Output: Local edge length $\hat{l}^T_{(c,r)}$ for the given 3×3 configuration.

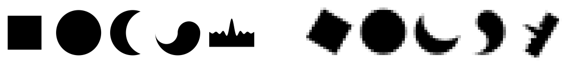
if $\tilde{p}_7 + \tilde{p}_8 + \tilde{p}_9 < \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3$ /* $y \geq kx + m$ */
  swap$(\tilde{p}_1, \tilde{p}_7)$
  swap$(\tilde{p}_2, \tilde{p}_8)$
  swap$(\tilde{p}_3, \tilde{p}_9)$
endif
if $\tilde{p}_3 + \tilde{p}_6 + \tilde{p}_9 < \tilde{p}_1 + \tilde{p}_4 + \tilde{p}_7$ /* $k < 0$ */
  swap$(\tilde{p}_1, \tilde{p}_3)$
  swap$(\tilde{p}_4, \tilde{p}_6)$
  swap$(\tilde{p}_7, \tilde{p}_9)$
endif
if $\tilde{p}_4 + \tilde{p}_7 + \tilde{p}_8 < \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_6$ /* $k > 1$ */
  swap$(\tilde{p}_2, \tilde{p}_4)$
  swap$(\tilde{p}_3, \tilde{p}_7)$
  swap$(\tilde{p}_6, \tilde{p}_8)$
endif
$\tilde{s}_1 = \tilde{p}_1 + \tilde{p}_4 + \tilde{p}_7$
$\tilde{s}_2 = \tilde{p}_2 + \tilde{p}_5 + \tilde{p}_8$
$\tilde{s}_3 = \tilde{p}_3 + \tilde{p}_6 + \tilde{p}_9$

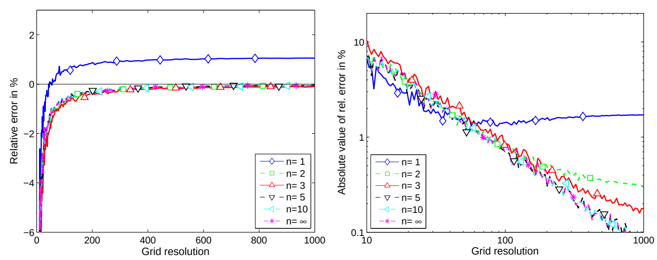
$\tilde{u}_l = (\tilde{s}_1 + \tilde{s}_2)/2$
$\tilde{u}_r = (\tilde{s}_2 + \tilde{s}_3)/2$
if $1 \leq \tilde{u}_l < 2$
  $\tilde{d}_l = \tilde{s}_2 - \tilde{s}_1$
  $\hat{l}_l = \frac{\gamma_n}{2}\sqrt{1 + \tilde{d}_l^2}$
else
  $\hat{l}_l = 0$
endif
if $1 < \tilde{u}_r \leq 2$
  $\tilde{d}_r = \tilde{s}_3 - \tilde{s}_2$
  $\hat{l}_r = \frac{\gamma_n}{2}\sqrt{1 + \tilde{d}_r^2}$
else
  $\hat{l}_r = 0$
endif
$\hat{l}^T_{(c,r)} = \hat{l}_l + \hat{l}_r$

Only **integer arithmetics** used locally (fast, exact).
Only **local** information is used (fast, stable, parallelizable).

Perimeter estimation - evaluation

Trade-off between spatial and grey-level resolution



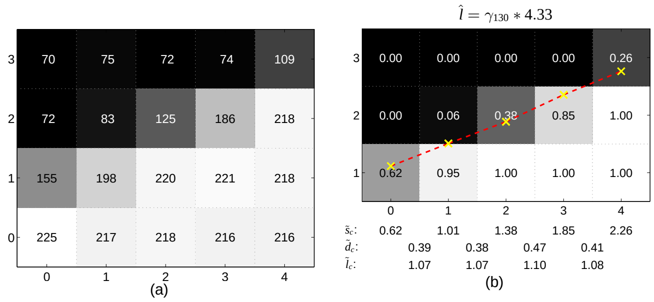


(a) lin-lin scale (b) log-log scale

Relative errors in percent for test shapes digitized at increasing resolution for 5 different quantization levels and non-quantized ($n = \infty$).

Segm. (soft threshold) + perimeter estimation

Digital photos of the straight edge of a white paper on a black background at a number of angles using a Panasonic DMC-FX01 digital camera in grey-scale mode.



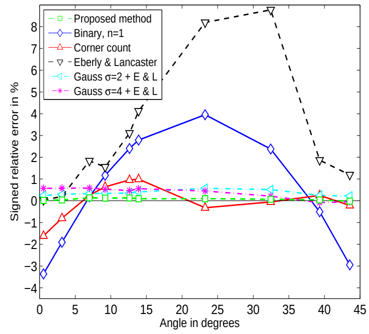
(a) Close up of the straight edge of a white paper imaged with a digital camera. (b) Segmentation output from Algorithm 2 using 130 positive grey-levels. Approximating edge segments are superimposed.

Results – Segm. method 2 + perimeter est.

The observed noise range in the images is between 20 and 50 grey-levels, out of 255, and the found value of n in the segmentation varies from 90 to 140 for the different photos.

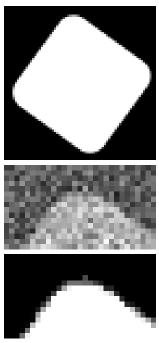
The observed maximal errors for the methods are as follows:

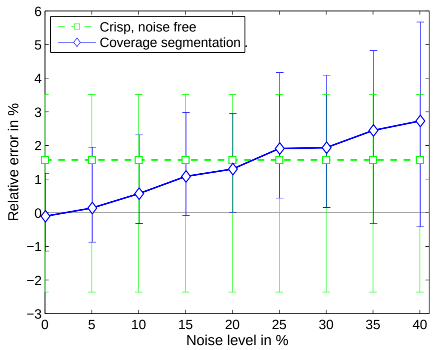
- Proposed method **0.14%**;
- Binary 3.95%;
- Corner count 1.61%;
- Eberly & Lancaster 8.78%;
- Gauss $\sigma = 2$ + E & L 0.57%;
- Gauss $\sigma = 4$ + E & L 0.58%.



Perimeter estimation error in a noisy environment

in combination with coverage segmentation



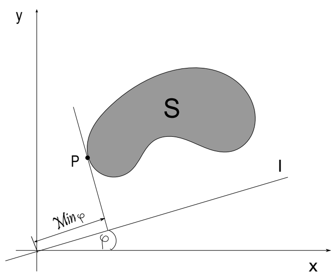


Left: (top) Synthetic test objects. (middle) Part of object with 30% noise added. (bottom) Coverage segmentation result for 30% noise. **Right:** Estimation errors for increasing levels of noise. Green is noise free crisp reference. Lines show averages for 50 observations and bars indicate max and min errors.

Orthogonal projection of a shape

Extremal points of a set S in the direction φ is defined as

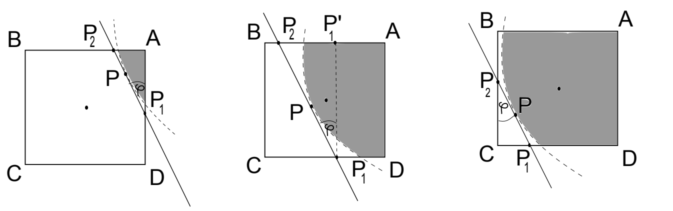
$$Min_\varphi(S) = \min_{(x,y)\in S}(x\cos\varphi + y\sin\varphi).$$



Tangent direction at extremal point P is orthogonal to the line $y = x\tan\varphi$

Orthogonal projection of a shape

Boundary of a smooth shape S in pixel A containing extremal point P, can be approximated by tangent line in extremal point.



Triangle
$P_1(i, j+1-\sqrt{2\cdot\alpha(i,j)\cdot\cot\varphi})$
$P_2(i-\sqrt{2\cdot\alpha(i,j)\cdot\tan\varphi}, j+1)$,

Trapeze
$P_1(i-\frac{2\cdot\alpha(i,j)-\tan\varphi}{2}, j)$
$P_2(i-\frac{2\cdot\alpha(i,j)+\tan\varphi}{2}, j+1)$,

Pentagon
$P_1(i-1+\sqrt{2\cdot(1-\alpha(i,j))\cdot\tan\varphi}, j)$
$P_2(i-1, j-\sqrt{2\cdot(1-\alpha(i,j))\cdot\cot\varphi})$.

Algorithm: Projection of an extremal point of a given shape onto a given direction.

- *Input: Pixels $p_{(i,j)}$, $i \in \{1,\dots m\}$, $j \in \{1,\dots n\}$, with values of area coverage digitization $\alpha(i,j)$ of given shape S, and direction φ for projecting.*
- *Output: Value of projection, Min_φ, of minimal extremal point onto the direction φ.*

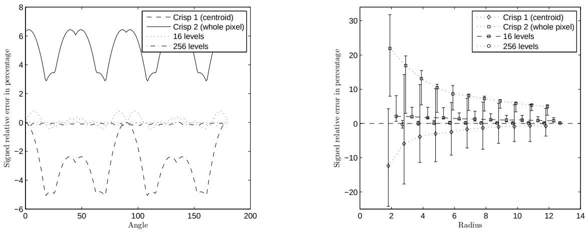
for $i \in \{1,\dots,m\}, j \in \{1,\dots,n\}, \alpha(i,j) > 0$

if $0 < \alpha(i,j) < \frac{\tan\varphi}{2}$
$\tilde{x}_i = i,$
$\tilde{y}_j = j + 1 - \sqrt{2\cdot\alpha(i,j)\cdot\cot\varphi}$
end if

if $\frac{\tan\varphi}{2} \le \alpha(i,j) \le \frac{1-\tan\varphi}{2}$
$\tilde{x}_i = i - \frac{2\cdot\alpha(i,j)-\tan\varphi}{2},$
$\tilde{y}_j = j$
end if

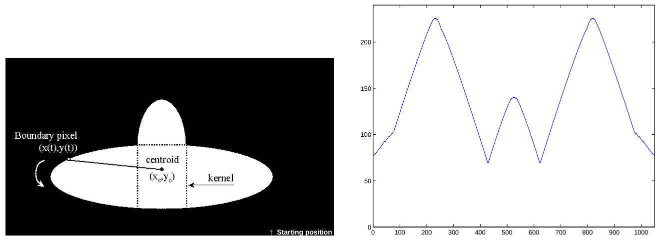
if $\alpha(i,j) > 1 - \frac{\tan\varphi}{2}$
$\tilde{x}_i = i - 1 + \sqrt{2\cdot(1-\alpha(i,j))\cdot\tan\varphi},$
$\tilde{y}_j = j$
end if
$\tilde{P}_{i,j} = \tilde{x}_i\cdot\cos\varphi + \tilde{y}_j\cdot\cos\varphi$
end for
$\tilde{Min}_\varphi = \min\{\tilde{P}_{i,j} \mid i \in \{1,\dots m\}, j \in \{1,\dots n\}\}.$

Diameter of a shape



Performance of estimation of a diameter estimator. Signed relative error of width of a circle, for different angles of projection and fixed radius (left) and for increasing radius of a disk (right)

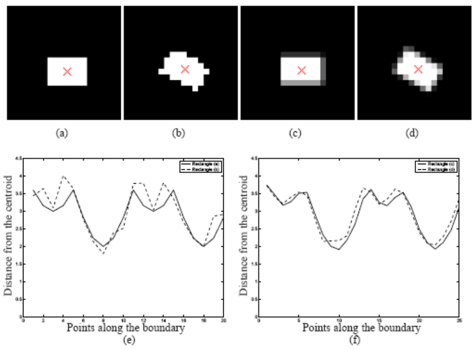
Signature of a shape – work in progress



A star-shaped object with respect to the centroid and its corresponding shape signature.

Signature of a shape: Sensitivity to rotation

Signature based on a Euclidean distance from a shape centroid computed from coverage representation of a square by fuzzification principle (averaging over α-cuts)



Work to do: further adjust the method to the coverage representation and derive estimation error bounds.

Conclusions

- Main results presented here are the given upper bounds for the estimation error as functions of spatial and coverage resolution.
- For some of the descriptors only statistical studies are performed. Results are encouraging, but should preferably be theoretically supported.
- We have confirmed that inter-relations between the two types of resolutions affect the precision of estimation, and that one of the resolutions can, to some extent, be used to compensate for the other.
- It is usually the case that spatial resolution is given by the imaging device and cannot be changed, whereas improved intensity information, or simply better utilization of grey-levels, already at hand, may be much more easily accessible.
- It is clear that there are still many more descriptors/estimators to address.