Probabilistic reasoning about simply typed lambda terms

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Abstract. Reasoning with uncertainty has gained an important role in computer science, artificial intelligence and cognitive science. These applications urge for development of formal models which capture reasoning of probabilistic features. We propose a *formal model* for reasoning about probabilities of simply typed lambda terms. We present its syntax, Kripke-style semantics and axiomatic system. The main results are the corresponding soundness and strong completeness, which rely on two key facts: the completeness of simple type assignment and the existence of the maximal consistent extension of a consistent set.

1 Introduction

In the last three decades several formal tools have been developed for reasoning about uncertain knowledge. One of these approaches concerns formalization in terms of probabilistic logics. Although the idea of probabilistic logic can be traced back to Leibnitz, Lambert and Boole, the modern development was started by Nils Nilsson, who tried to provide a logical framework for uncertain reasoning [22]. After Nilsson, a number of researchers proposed formal systems for probabilistic reasoning, for example [9], [10]. The general lack of compactness for probabilistic logics causes that one of the main proof-theoretical problems in this framework is to provide a strongly complete axiomatic system. Several infinitary logics have been introduce to deal with that issue, a detailed overview can be found in [25], [16], and [24]. Note that the term infinitary concerns the meta language only, i.e. the object language is countable, and formulas are finite, while only proofs are allowed to be infinite. It turns out that this approach can be combined, for example, with temporal [7], [23] and intuitionistic reasoning [20]. So, building on our previous experience (e.g. see [17], [24], [26]), we describe a class of so called measurable models for probabilistic extension of simple type assignment and give a sound and strongly complete (every consistent set of formulas is satisfiable) infinitary axiomatization.

The λ-calculus, proposed by Church in the early 1930s, is a simple formal system capable of expressing all effectively computable functions, and equivalent to Turing machines. Typed λ-calculus is a restricted system, where application is controlled by objects (types) assigned to λ-terms. Already Church introduced the system with simple (functional) types that turned out to represent the computational interpretation of intuitionistic natural deduction as stated by the well-known Curry-Howard correspondence. A variety of type systems (such as intersection types, dependent types, polymorphic types, etc...) were proposed in the last decades, finding application in programming languages for certified compilers, automated theorem provers and proof assistants, software verification, computational linguistics, among others. For more details we refer the reader to [3], [4], [11]. Soundness and completeness of the simple type assignment has been proved with respect to semantics developed in [13] and [14].

Reasoning with uncertainty has gained an important role in computer science, artificial intelligence and cognitive science. These applications urge for development of formal models which capture reasoning of probabilistic features [12]. This is our motivation for developing a new formal model for reasoning about simply typed lambda terms.

Contributions and main results. We introduce in this paper a formal model $\mathsf{P}\Lambda_{\rightarrow}$ for reasoning about probabilities of simply typed lambda terms which is a combination of lambda calculus and probabilistic logic. We propose its syntax, Kripke-style semantics and an infinitary axiomatization.

We first endow the language of typed lambda calculus with a probabilistic operator $P_{\geq s}$ and obtain formulas of the form

$$P_{\geq s}M : \sigma$$

to express that the probability that the lambda term M is of type σ is equal to or greater than s. More generally, formulas are of the form $P_{\geq s}\alpha$, where α is typed lambda statement $M : \sigma$ or its Boolean combination. We then propose a semantics of $\mathsf{P}\Lambda_{\rightarrow}$ based on a set of possible worlds, where each possible world is a lambda model. The set of possible worlds is equipped with a probability measure μ. The set $[\alpha]$ is the set of possible worlds that satisfy the formula α. Then the probability of α is obtained as $\mu([\alpha])$. Finally, we give an infinitary axiomatization of $\mathsf{P}\Lambda_{\rightarrow}$ and prove the deduction theorem.

The main results are the soundness and strong completeness of $\mathsf{P}\Lambda_{\rightarrow}$ with respect to the proposed model, where strong completeness means that every consistent set of formulas is satisfiable. The construction of the canonical model is crucial for the proof and relays on two key facts. The first one is that the simple type assignment is complete with respect to the simple semantics and the second one is the property that every consistent set can be extended to the maximal consistent set.

Related work In the last decade, several probabilistic extensions of the λ-calculus have been introduced and investigated. They are concerned with introducing non-determinism and probabilities into the syntax and operational semantics of the λ-calculus in order to formalize computation in the presence of uncertainty rather than with providing a framework that would enable probabilistic reasoning about typed terms and type assignments.

Audebaud and Paulin-Mohring in [2] gave the axiomatic rules for the estimation of the probability that programs satisfy some given properties. Furthermore, Dal Lago and Zorzi in [19] considered a non deterministic extension of lambda calculus, defined small-step and big-step semantics, and proved that calculus is sound and complete with respect to computable probability distributions, whereas Bizjak and Birkedal in [5] constructed a step-indexed logical relations for a probabilistic extension of a certain higher-order programming language and showed that the relation is sound and complete with respect to the contextual preorder. Ehrhard et al. in [8] study the probabilistic coherent spaces as a denotational semantics and show soundness of a probabilistic extension of the untyped λ-calculus, which is a quantitative refinement to the soundness of the untyped λ-calculus with respect to the Scott's model in a probabilistic setting.

A slightly similar approach to ours, that provides a framework for probabilistic reasoning about typed terms, was treated by Cooper et al. in [6], where the authors proposed a probabilistic type theory in order to formalize computation with statements of the form "a given type is assigned to a given situation with probability p". However, the developed theory was used for analyzing semantic learning of natural languages in the domain of computational linguistics, and no soundness or completeness issues were discussed.

We provide a formal model for probabilistic reasoning about typed lambda terms. Our formal model is developed along the lines of the method that was used in [18] and [17] to obtain a formal model for probabilistic justification logic. However, the logic of uncertain justification already existed ([21]), so the authors have compared in [17] their logic with Milnikel's logic proposed in [21], whereas, to the best of our knowledge, the formal model that we propose is the first one.

Outline of the paper. Section 2 revisits basic notions of lambda calculus, simple type assignment and simple semantics. In Section 3 we present the syntax and Kripke-style semantics of our probabilistic formal model for reasoning about simply typed lambda terms. The axiomatic system, together with the soundness theorem, is given in Section 4. The completeness of the proposed probabilistic formal model is proved in Section 5.

2 Simple type assignment $\mathbf{\Lambda}_{\rightarrow}$

In this section, we recall some basic notions of lambda calculus ([3]), simple types ([14], [4]), lambda models ([3], [14], [15]) and revisit the soundness and completeness result for the simple type assignment proved in [13].

2.1 Lambda terms and types

We recall now some basic notions of the simply typed lambda calculus.

Let $\mathtt{V}_\Lambda = \{x, y, z, \ldots, x_1, \ldots\}$ be a countable set of λ-term variables. *Terms (λ-terms)* are generated by the following grammar:

$$M ::= x \mid \lambda x.M \mid MM.$$

The set of all terms is denoted by Λ and is ranged over by $M, N, \ldots, M_1, \ldots$. The operator λx is a binder and the set of *free variables* of a term M is defined as usual. The α-conversion, the renaming of bound variables, enables to implement Barendregt's convention that bound variables are distinct from free variables.

The β-reduction is a rewriting rule $(\lambda x.M)N \to_\beta M[N/x]$. The definition and main properties of β-reduction (and $\beta\eta$-reduction) can be found in [3], [14]. The lambda term M is β-equal to the lambda term N (notion $M =_\beta N$) if and only if there is a sequence $M \equiv N_0, N_1, \ldots, N_n \equiv N$, where $N_i \to_\beta N_{i+1}$ or $N_{i+1} \to_\beta N_i$ for all $i \in \{0, 1, \ldots, n\}$.

Let $\mathtt{V}_{\mathtt{Type}} = \{a, b, c, \ldots\}$ be a denumerable set of propositional variables. *Types (simple types)* are generated by the following grammar:

$$\sigma ::= a \mid \sigma \to \sigma.$$

The set of all types is denoted by $\mathtt{Type}$ and is ranged over by $\sigma, \tau, \ldots, \sigma_1, \ldots$.

A *lambda statement* is an expression of the form $M : \sigma$, where $M \in \Lambda$ and $\sigma \in \mathtt{Type}$. Moreover, $x : \sigma$ is a *basic statement*. A *basis (context)* is a set of basic statements with distinct term variables (can be infinite).

Definition 1. *The simple type assignment, $\Lambda_\to$, is defined as follows:*

$$\frac{M : \sigma \to \tau \qquad N : \sigma}{MN : \tau}\ (\to_E)$$

$$\begin{array}{c}[x : \sigma]\\ \vdots\\ \dfrac{M : \tau}{\lambda x.M : \sigma \to \tau}\ (\to_I)\end{array}$$

$$\frac{M : \sigma \qquad M =_\beta N}{N : \sigma}\ (\mathrm{eq})$$

If $M : \sigma$ is derivable by the given rules from a basis Γ, it is denoted by $\Gamma \vdash M : \sigma$. In the sequel we work with this simple type assignment *à la* Curry. There is an equivalent simple type assignment *à la* Church, which is out of our scope.

2.2 Lambda models

We assume that the reader is familiar with the notion of the lambda model and the *interpretation of terms* in it. Basic notions and definitions can be found in [3], [13], [14], [15].

Term model $\mathcal{M} = \langle D, \cdot, [\![\ ]\!] \rangle$ is defined as follows:

Definition 2. *(i) Domain of a term model is a set of all convertibility-classes of terms. For $M \in \Lambda$, the convertibility-class represented by M will be denoted by $[M]$, i.e., $[M] = \{N : N =_\beta M\}$.*

(ii) If $\rho : \mathtt{V}_\Lambda \to D$ is the valuation of term variables in D, then $[\![M]\!]_\rho \in D$ is the interpretation of $M \in \Lambda$ in $\mathcal{M}$ via ρ.

(iii) Map $\cdot$ is defined by

$$[M] \cdot [N] = [MN],$$

and $[\![\ ]\!]_\rho$ is defined by

$$[\![M]\!]_\rho = [M[N_1, \ldots, N_n/x_1, \ldots, x_n]],$$

where $x_1, \ldots, x_n$ are the free variables of M, and $\rho(x_i) = [N_i]$ and $[\cdots/\cdots]$ is simultaneous substitution.

(iv) Let $\xi : \mathtt{V}_{\mathtt{Type}} \to \mathcal{P}(D)$ be a valuation of type variables. The interpretation of $\sigma \in \mathtt{Type}$ in $\mathcal{M}$ via ξ, denoted by $[\![\sigma]\!]_\xi \in \mathcal{P}(D)$, is defined:

- $[\![a]\!]_\xi = \xi(a)$;
- $[\![\sigma \to \tau]\!]_\xi = \{d \in D \mid \forall e \in [\![\sigma]\!]_\xi,\ d \cdot e \in [\![\tau]\!]_\xi\}$.

(v)
- $\mathcal{M}, \rho, \xi \models M : \sigma$ *iff* $[\![M]\!]_\rho \in [\![\sigma]\!]_\xi$;
- $\mathcal{M}, \rho, \xi \models \Gamma$ *iff* $\mathcal{M}, \rho, \xi \models x : \sigma$ *for all* $x : \sigma \in \Gamma$;
- $\Gamma \models M : \sigma$ *iff* $(\forall \mathcal{M}, \rho, \xi \models \Gamma)\ \mathcal{M}, \rho, \xi \models M : \sigma$.

The soundness and completeness of type assignment is proved in [13] with this notion of lambda model. The above semantics is called *simple semantics*. The following results are the key for proving strong completeness for the logic we propose in this paper.

Theorem 1 (Soundness). $\Gamma \vdash M : \sigma \ \Rightarrow\ \Gamma \models M : \sigma$.

Theorem 2 (Completeness). $\Gamma \models M : \sigma \Rightarrow \Gamma \vdash M : \sigma$.

3 Probabilistic logical system for simply typed lambda terms $\mathsf{P}\Lambda_\to$

The probabilistic logical system for typed lambda terms, $\mathsf{P}\Lambda_\to$, is a probabilistic logic over the simple type assignment $\Lambda_\to$. In this section, we introduce the syntax and semantics of $\mathsf{P}\Lambda_\to$.

3.1 Syntax of $\mathsf{P}\Lambda_\to$

Let S be the set of rational numbers from $[0, 1]$, i.e., $\mathsf{S} = [0, 1] \cap \mathbb{Q}$. The *alphabet* of the logic $\mathsf{P}\Lambda_\to$ consists of

- all symbols needed to define simply typed lambda terms, given in 2.1,
- the classical propositional connectives $\neg$ and $\wedge$,
- the list of probability operators $P_{\geq s}$, for every $s \in \mathsf{S}$.

Other propositional connectives $\Rightarrow$, $\vee$, $\Leftrightarrow$ are defined as usual.

Basic formulas. All lambda statements of the form $M : \sigma$, where $M \in \Lambda$ and $\sigma \in \mathtt{Type}$, or statements of the same form connected with Boolean connectives, will be called *basic formulas*. Basic formulas are generated by the following grammar:

$$\mathsf{For_B} \quad \alpha ::= M : \sigma \mid \alpha \wedge \alpha \mid \neg\alpha.$$

The set of all basic formulas is denoted by $\mathsf{For_B}$ and will be ranged over by $\alpha, \beta, \ldots$, possibly indexed.

Probabilistic formulas. If $\alpha \in \mathsf{For_B}$ and $s \in \mathsf{S}$, then a *basic probabilistic formula* is any formula of the form $P_{\geq s}\alpha$. The set of all probabilistic formulas, denoted by $\mathsf{For_P}$, is the smallest set containing all basic probabilistic formulas which is closed under Boolean connectives.

Probabilistic formulas are generated by the following grammar:

$$\mathsf{For_P} \quad \phi ::= P_{\geq s}\alpha \mid \phi \wedge \phi \mid \neg\phi.$$

The set $\mathsf{For_P}$ will be ranged over by ϕ, $\psi, \ldots$, possibly with subscripts.

Formulas of $\mathsf{P}\Lambda_{\rightarrow}$*.* The language of $\mathsf{P}\Lambda_{\rightarrow}$ consists of both basic formulas and probabilistic formulas

$$\mathsf{For}_{\mathsf{P}\Lambda_{\rightarrow}} = \mathsf{For_B} \cup \mathsf{For_P}.$$

The set of formulas $\mathsf{For}_{\mathsf{P}\Lambda_{\rightarrow}}$ will be ranged over by $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2, \ldots$.

We use the following abbreviations to introduce other inequalities:

$$\begin{aligned}
&P_{<s}\alpha \text{ stands for } \neg P_{\geq s}\alpha,\\
&P_{\leq s}\alpha \text{ stands for } P_{\geq 1-s}\neg\alpha,\\
&P_{>s}\alpha \text{ stands for } \neg P_{\leq s}\alpha,\\
&P_{=s}\alpha \text{ stands for } P_{\geq s}\alpha \wedge \neg P_{>s}\alpha.
\end{aligned}$$

We also denote both $\alpha \wedge \neg\alpha$ and $\phi \wedge \neg\phi$ by $\bot$ (and dually for $\top$).

Note that neither mixing of basic formulas and probabilistic formulas, nor nested probability operators is allowed.

For example, the following two expressions are *not* (well defined) formulas of the logic $\mathsf{P}\Lambda_{\rightarrow}$:

$$\alpha \wedge P_{\geq \frac{1}{2}}\beta, \qquad P_{\geq \frac{1}{3}}P_{\geq \frac{1}{2}}\alpha.$$

The former is not well defined since it is a Boolean combination of a basic formula and probabilistic formula, whereas the latter is not well defined $\mathsf{P}\Lambda_{\rightarrow}$ formula because it contains nested probability operators.

3.2 Semantics of $\mathsf{P}\Lambda_{\rightarrow}$

The semantics for $\mathsf{P}\Lambda_{\rightarrow}$ is a Kripke-style semantics based on the possible-world approach.

Definition 3 **($\mathsf{P}\Lambda_{\rightarrow}$-structure).** *A $\mathsf{P}\Lambda_{\rightarrow}$-structure is a tuple $\mathcal{M} = \langle W, \rho, \xi, H, \mu \rangle$, where:*

(i) W *is a nonempty set of worlds, where each world is one lambda model, i.e., for every* $w \in W$, $w = \langle \mathcal{L}(w), \cdot_w, [\![\ ]\!]_w \rangle$;

(ii) $\rho : \mathsf{V}_\Lambda \times \{w\} \longrightarrow \mathcal{L}(w)$, $w \in W$;

(iii) $\xi : \mathsf{V}_{\mathsf{Type}} \times \{w\} \longrightarrow \mathcal{P}(\mathcal{L}(w))$, $w \in W$;

(iv) H *is an algebra of subsets of* W, *i.e.* $H \subseteq \mathcal{P}(W)$ *such that*
- $W \in H$,
- *if* $U, V \in H$, *then* $W \setminus U \in H$ *and* $U \cup V \in H$;

(v) μ *is a finitely additive probability measure defined on* H, *i.e.,*
- $\mu(W) = 1$,
- *if* $U \cap V = \emptyset$, *then* $\mu(U \cup V) = \mu(U) + \mu(V)$, *for all* $U, V \in H$.

The elements of H are called *measurable worlds*. We will write $\rho_w(x)$, instead of $\rho(x, w)$ and similarly for ξ.
We say that a lambda statement $M : \sigma$ holds in a world w, notation $w \models M : \sigma$, if and only if

$$[\![M]\!]^w_\rho \in [\![\sigma]\!]^w_\xi,$$

where $[\![M]\!]^w_\rho$ is the interpretation of a term M in a world w via ρ, and $[\![\sigma]\!]^w_\xi$ is the interpretation of a type σ in a world w via ξ. Also we define that
$w \models M : \sigma \wedge N : \tau$ iff $w \models M : \sigma$ and $w \models N : \tau$,
$w \models \neg(M : \sigma)$ iff $w \not\models M : \sigma$.

For a given $\alpha \in \mathsf{For_B}$ and $\mathsf{P}\Lambda_\rightarrow$-structure $\mathcal{M}$, let

$$[\alpha]_\mathcal{M} = \{w \in W \mid w \models \alpha\}.$$

We will omit the subscript $\mathcal{M}$ when there is no ambiguity from the context.

Definition 4 (Measurable structure). *A structure* $\mathcal{M}$ *is measurable if* $[\alpha]_\mathcal{M} \in H$ *for every* $\alpha \in \mathsf{For_B}$. *The class of all measurable structures of the logic* $\mathsf{P}\Lambda_\rightarrow$ *will be denoted by* $\mathsf{P}\Lambda_\rightarrow^{\mathsf{Meas}}$.

Definition 5 (Satisfiability relation). *The satisfiability relation* $\models \subseteq \mathsf{P}\Lambda_\rightarrow^{\mathsf{Meas}} \times \mathsf{For}_{\mathsf{P}\Lambda_\rightarrow}$ *is defined in the following way:*

- $\mathcal{M} \models M : \sigma$ *iff* $w \models M : \sigma$, *for all* $w \in W$;
- $\mathcal{M} \models P_{\geq s}\alpha$ *iff* $\mu([\alpha]) \geq s$;
- $\mathcal{M} \models \neg\mathfrak{A}$ *iff it is not the case that* $\mathcal{M} \models \mathfrak{A}$;
- $\mathcal{M} \models \mathfrak{A}_1 \wedge \mathfrak{A}_2$ *iff* $\mathcal{M} \models \mathfrak{A}_1$ *and* $\mathcal{M} \models \mathfrak{A}_2$.

Definition 6 (Formula satisfiability). *Let* $\mathfrak{A} \in \mathsf{For}_{\mathsf{P}\Lambda_\rightarrow}$ *be a formula and* $F \subseteq \mathsf{For}_{\mathsf{P}\Lambda_\rightarrow}$

- $\mathfrak{A}$ *is satisfiable if there is an* $\mathsf{P}\Lambda_\rightarrow^{\mathsf{Meas}}$*-model* $\mathcal{M}$ *such that* $\mathcal{M} \models \mathfrak{A}$;
- $\mathfrak{A}$ *is valid if for every* $\mathsf{P}\Lambda_\rightarrow^{\mathsf{Meas}}$*-model* $\mathcal{M}$, $\mathcal{M} \models \mathfrak{A}$;
- *A set of formulas* F *is satisfiable if there is a* $\mathsf{P}\Lambda_\rightarrow^{\mathsf{Meas}}$*-model* $\mathcal{M}$ *such that* $\mathcal{M} \models \mathfrak{A}$ *for every* $\mathfrak{A} \in F$.

We now give a couple of simple examples in order to clarify the above notions.

Example 1. Consider the following model with three worlds, i.e., let $\mathcal{M} = \langle W, \rho, \xi, H, \mu \rangle$, where:

- $W = \{w_1, w_2, w_3\}$,
- $H = \mathcal{P}(W)$,
- $\mu(\{w_j\}) = \frac{1}{3}$, $j = 1, 2, 3$,

and ρ and ξ are defined such that
$w_1 \models (x : \sigma \to \tau) \wedge (y : \sigma)$,
$w_2 \models (x : \sigma_1 \to \tau) \wedge (y : \sigma_1)$,
$w_3 \models (x : \sigma_2 \to \tau) \wedge (y : \sigma_2)$ (Figure 1).
It is obvious that $\mathcal{M} \models P_{=\frac{1}{3}}(x : \sigma \to \tau)$, $\mathcal{M} \models P_{=\frac{1}{3}}(y : \sigma)$ and $\mathcal{M} \models P_{=1}(xy : \tau)$. Note that this example shows that in the case of an application of two terms, the probability of an application can not be smaller than the probability of the conjunction of its components, but can be any number greater than or equal to it (and less or equal to 1).

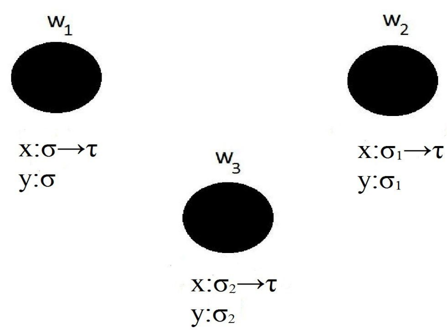

Fig. 1.

The previous example showed that an application always has a probability bigger than or equal to the conjuction of its components. On the other hand, each conjuct can have a probability greater than the application as we will show in the following example.

Example 2. Let $\mathcal{M} = \langle W, \rho, \xi, H, \mu \rangle$, where:

- $W = \{w_1, w_2, w_3\}$,
- $H = \mathcal{P}(W)$,
- $\mu(\{w_j\}) = \frac{1}{3}$, $j = 1, 2, 3$,

and ρ and ξ are defined such that
$w_1 \models (x : \sigma \to \tau) \wedge (y : \sigma)$,

$w_2 \models (x : \sigma \to \tau) \wedge (y : \sigma_1)$,
$w_3 \models (x : \sigma_2 \to \tau) \wedge (y : \sigma)$.
Now, it is clear that $\mathcal{M} \models P_{=\frac{1}{3}}(xy : \tau)$, while $\mathcal{M} \models P_{=\frac{2}{3}}(x : \sigma \to \tau)$ and $\mathcal{M} \models P_{=\frac{2}{3}}(y : \sigma)$.

The next example shows that we must provide an infinitary axiomatization in order to obtain strong completeness for our formal model.

Example 3. Consider the set

$$F = \{\neg P_{=0}\alpha\} \cup \{P_{<\frac{1}{n}}\alpha \mid n \quad is \quad a \quad positive \quad integer\}.$$

Every finite subset of F is clearly $\mathsf{P}\Lambda_{\to}^{\mathsf{Meas}}$-satisfiable, but the set F itself is not, since there is no real number greater than 0 and smaller than all positive rationals due to the Archimedean property of real numbers [4]. Therefore, the compactness theorem which states that "if every finite subset of F is satisfiable, then F is satisfiable" does not hold for $\mathsf{P}\Lambda_{\to}$.

4 The axiomatization $Ax_{\mathsf{P}\Lambda_{\to}}$

We introduce an axiomatic system for the logic $\mathsf{P}\Lambda_{\to}$ which will be denoted by $Ax_{\mathsf{P}\Lambda_{\to}}$. Inference rules will be divided in two groups, such that inference rules from the first group can be applied only on lambda statements.

Axiom schemes

(1) all instances of the classical propositional tautologies, (atoms are any $\mathsf{P}\Lambda_{\to}$-formulas),
(2) $P_{\geq 0}\alpha$,
(3) $P_{\leq r}\alpha \Rightarrow P_{<s}\alpha$, $s > r$,
(4) $P_{<s}\alpha \Rightarrow P_{\leq s}\alpha$,
(5) $(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\geq 1}(\neg\alpha \vee \neg\beta)) \Rightarrow P_{\geq min\{1,r+s\}}(\alpha \vee \beta)$,
(6) $(P_{\leq r}\alpha \wedge P_{<s}\beta) \Rightarrow P_{<r+s}(\alpha \vee \beta)$, $r + s \leq 1$,
(7) $P_{\geq 1}(\alpha \Rightarrow \beta) \Rightarrow (P_{\geq s}\alpha \Rightarrow P_{\geq s}\beta)$.

Inference Rules I

$$(1) \qquad \frac{M : \sigma \to \tau \qquad N : \sigma}{MN : \tau} \ (\to_E)$$

$$(2) \qquad \frac{\begin{matrix}[x : \sigma] \\ \vdots \\ M : \tau\end{matrix}}{\lambda x.M : \sigma \to \tau} \ (\to_I)$$

[4] For any real number $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

$$(3) \quad \frac{M:\sigma \qquad M =_\beta N}{N:\sigma} \text{ (eq)}$$

Inference Rules II

(1) From $\mathfrak{A}_1$ and $\mathfrak{A}_1 \Rightarrow \mathfrak{A}_2$ infer $\mathfrak{A}_2$,
(2) from α infer $P_{\geq 1}\alpha$,
(3) from the set of premises

$$\{\phi \Rightarrow P_{\geq s-\frac{1}{k}}\alpha \mid k \geq \frac{1}{s}\}$$

infer $\phi \Rightarrow P_{\geq s}\alpha$.

Axiom 2 announces that every formula is satisfied in a set of worlds whose measure is at least 0, and we can easily infer (using $\neg\alpha$ instead of α) that the upper bound is 1, i.e., $P_{\leq 1}\alpha$. Axioms 3 and 4 provide the monotonicity of a measure, Axioms 5 and 6 correspond to the finite additivity of a measure, whereas Axiom 7 ensures that equivalent formulas have equal measures.

Inference Rules I are the rules that correspond to correct inference of typed lambda terms.

Inference Rules II:

- Rule II.1 is modus ponens (MP);
- Rule II.2 is the probability necessitation;
- Rule II.3 is the only infinitary rule of inference, and states that if probability is arbitrary close to s then it is at least s.

Definition 7 (Inference relation).
Let T be a set of formulas and $\mathfrak{A}$ a formula.

1. *$T \vdash \mathfrak{A}$ means that there exists a sequence $\mathfrak{A}_0, \dots, \mathfrak{A}_{\lambda+1}$ (λ is finite or countable ordinal) of formulas, such that $\mathfrak{A}_{\lambda+1} = \mathfrak{A}$ and for all $i \leqslant \lambda + 1$, $\mathfrak{A}_i$ is an axiom-instance, or $\mathfrak{A}_i \in T$, or $\mathfrak{A}_i$ can be derived by some inference rule applied on some previous members of the sequence.*
2. *Instead of $\emptyset \vdash \mathfrak{A}$ we write $\vdash \mathfrak{A}$. Any formula $\mathfrak{A}$ such that $\vdash \mathfrak{A}$ will be called a theorem.*
3. *T is* consistent *if*
   - *there is at least a formula $\alpha \in \mathsf{For_B}$ and a formula $\phi \in \mathsf{For_P}$ that are not deducible from T and*
   - *let $\{x_i, \mid i \in I\}$ be the set of all free variables of lambda terms that appear as a (sub)formulas in T. Then all basic statements of the form $x_i : \tau_i$ (for appropriate types τ_i) are also in T.*

   Otherwise, T is inconsistent.
4. *T is a maximally consistent set if it is consistent and:*
   (1) for every $\alpha \in \mathsf{For_B}$, if $T \vdash \alpha$, then $\alpha \in T$ and $P_{\geq 1}\alpha \in T$,
   (2) for every $\phi \in \mathsf{For_P}$, either $\phi \in T$ or $\neg\phi \in T$.
5. *T is deductively closed if for every $\mathfrak{A} \in \mathsf{For_{P\Lambda_\rightarrow}}$, if $T \vdash \mathfrak{A}$, then $\mathfrak{A} \in T$.*

Note that it is not required that for every $\alpha \in \mathsf{For_B}$, either α or $\neg\alpha$ belongs to a maximal consistent set (as it is done for formulas from $\mathsf{For_P}$). It can be proved that, otherwise, in our canonical model, for each α we would have $P_{=1}\alpha$ or $P_{=0}\alpha$, so the probability operator would not make sense.

Theorem 3 (Deduction theorem). *Let T be a set of formulas and $\phi, \psi \in \mathsf{For_P}$. If $T \cup \{\phi\} \vdash \psi$ then $T \vdash \phi \Rightarrow \psi$.*

Theorem 4 (Soundness). *The axiomatic system $Ax_{\mathsf{P}\wedge_{\rightarrow}}$ is sound with respect to the class of $\mathsf{P}\wedge_{\rightarrow}^{\mathsf{Meas}}$-models.*

5 Completeness

In order to prove the completeness theorem we start with some auxiliary statements. After that, we show how to extend a consistent set of formulas T to a maximal consistent set of formulas $T^\star$. Finally, we construct the canonical model using the set $T^\star$ such that $\mathcal{M}_{T^\star} \models \mathfrak{A}$ iff $\mathfrak{A} \in T^\star$.

Lemma 1. *Let T be a consistent set of formulas.*

(1) For any formula $\phi \in \mathsf{For_P}$, either $T \cup \{\phi\}$ is consistent or $T \cup \{\neg\phi\}$ is consistent.
(2) If $\neg(\phi \Rightarrow P_{\geq s}\alpha) \in T$, then there is some $n > \frac{1}{s}$ such that $T \cup \{\phi \Rightarrow \neg P_{\geq s - \frac{1}{n}}\alpha\}$ is consistent.

Lemma 2. *Let T be a maximal consistent set of formulas.*

(1) $\psi \in \mathsf{For_P}$, if $T \vdash \psi$, then $\psi \in T$.
(2) For any formula α, if $t = sup\{s \mid P_{\geq s}\alpha \in T\}$, and $t \in S$, then $P_{\geq t}\alpha \in T$.

Theorem 5. *Every consistent set can be extended to a maximal consistent set.*

Proof. Consider a consistent set T. By $\mathsf{Cn_B}(T)$ we will denote the *consistent* set of all basic formulas that are consequences of T. Let $\phi_0, \phi_1, \ldots$ be an enumeration of all formulas from $\mathsf{For_P}$. We define a sequence of sets T_i, $i = 0, 1, 2, \ldots$ as follows:

(1) $T_0 = T \cup \mathsf{Cn_B}(T) \cup \{P_{\geq 1}\alpha \mid \alpha \in \mathsf{Cn_B}(T)\}$,
(2) for every $i \geq 0$,
  (a) if $T_i \cup \{\phi_i\}$ is consistent, then $T_{i+1} = T_i \cup \{\phi_i\}$, otherwise
  (b) if ϕ_i is of the form $\psi \Rightarrow P_{\geq s}\beta$, then
  $T_{i+1} = T_i \cup \{\neg\phi_i, \psi \Rightarrow \neg P_{\geq s - \frac{1}{n}}\beta\}$, for some positive integer n, so that T_{i+1} is consistent, otherwise,
  (c) $T_{i+1} = T_i \cup \{\neg\phi_i\}$,
(3) $T^\star = \bigcup_{i=0}^{\infty} T_i$.

The set T_0 is obviously consistent. Note that the existence of the natural number (n) from the step $2(b)$ of the construction is provided by Lemma 1.(2), and each T_i is consistent.

It still remains to show that $T^\star$ is the maximal consistent set. The steps (1) and (2) of the above construction ensure that $T^\star$ is maximal.

$T^\star$ obviously does not contain all formulas. If $\alpha \in \mathsf{For_B}$, by the construction of T_0, α and $\neg\alpha$ can not be both in T_0. For a formula $\phi \in \mathsf{For_P}$, the set $T^\star$ does not contain both $\phi = \phi_i$ and $\neg\phi = \phi_j$, because the set $T_{\max\{i,j\}+1}$ is consistent.

Let us prove that $T^\star$ is deductively closed. If a formula $\alpha \in \mathsf{For_B}$ and $T \vdash \alpha$, then by the construction of T_0, $\alpha \in T^\star$ and $P_{\geq 1}\alpha \in T^\star$. Let $\phi \in \mathsf{For_P}$. It can be easily proved (induction on the length of the inference) that if $T^\star \vdash \phi$, then $\phi \in T^\star$. Note the fact that if $\phi = \phi_j$ and $T_i \vdash \phi$ it has to be $\phi \in T^\star$ because $T_{\max\{i,j\}+1}$ is consistent. Suppose that the sequence $\phi_1, \phi_2, \dots, \phi$ is the proof of ϕ from $T^\star$. If the mentioned sequence is finite, there must be some set T_i such that $T_i \vdash \phi$, and $\phi \in T^\star$. Therefore, suppose that the sequence is countably infinite. We can show that, for every i, if ϕ_i is obtained by an application of an arbitrary inference rule, and all the premises belong to $T^\star$, then, also $\phi_i \in T^\star$. If the inference rule is a finitary one, then there must be a set T_j which contains all the premises and $T_j \vdash \phi_i$. So, we conclude that $\phi_i \in T^\star$. Now, consider the infinitary Rule II.3. Let $\phi_i = \psi \Rightarrow P_{\geq s}\alpha$ be obtained from the set of premises $\{\phi_i^k = \psi \Rightarrow P_{\geq s_k}\alpha \mid s_k \in S\}$. By the induction hypothesis, we have that $\phi_i^k \in T^\star$, for every k. If $\phi_i \notin T^\star$, by step (2)(b) of the construction, there are some l and j so that $\neg(\psi \Rightarrow P_{\geq s}\alpha)$, $\psi \Rightarrow \neg P_{\geq s-\frac{1}{l}}\alpha \in T_j$. Thus, we have that for some $j' \geq j$:

- $\psi \wedge \neg P_{\geq s}\alpha \in T_{j'}$,
- $\psi \in T_{j'}$,
- $\neg P_{\geq s-\frac{1}{l}}\alpha \in T_{j'}$,
- $P_{\geq s-\frac{1}{l}}\alpha \in T_{j'}$ Ind. Hyp.

Contradiction with the consistency of a set $T_{j'}$.

Thus, $T^\star$ is a deductively closed set which does not contain all formulas, so it is consistent. □

Definition 8. *If $T^\star$ is the maximally consistent set of formulas, then a tuple $\mathcal{M}_{T^\star} = \langle W, \rho, \xi, H, \mu\rangle$ is defined:*

- $W = \{w = \langle \mathcal{L}(w), \cdot_w, [\![\ ]\!]_w\rangle \mid w \models Cn_B(T)\}$ *contains all term models that satisfy the set* $Cn_B(T)$,
- $\rho_w(x) = [x]$,
- $\xi_w(a) = \{[M] \in \mathcal{L}(w) \mid w \models M : a\}$,
- $H = \{[\alpha] \mid \alpha \in \mathsf{For_B}\}$, *where* $[\alpha] = \{w \in W \mid w \models \alpha\}$,
- $\mu([\alpha]) = \sup\{s \mid P_{\geq s}\alpha \in T^\star\}$.

Lemma 3. *(1) H is an algebra of subsets of W,*
(2) If $[\alpha] = [\beta]$, then $\mu([\alpha]) = \mu([\beta])$,
(3) $\mu([\alpha]) \geq 0$,
(4) $\mu(W) = 1$, $\mu(\emptyset) = 0$,

(5) $\mu([\alpha]) = 1 - \mu([\neg\alpha])$,
(6) $\mu([\alpha] \cup [\beta]) = \mu([\alpha]) + \mu([\beta])$, *for* $[\alpha] \cap [\beta] = \emptyset$.

Consequence of this Lemma is that $\mathcal{M}_{T^\star}$ is well defined.

Lemma 4. *Let* $T^\star$ *be a maximal consistent set of formulas. Then,* $\mathcal{M}_{T^\star} \in \mathsf{P\Lambda}_{\rightarrow}^{\mathsf{Meas}}$.

Proof. Directly from the construction of $\mathcal{M}_{T^\star}$. □

We are now ready to prove the main result of this paper.

Theorem 6 (Strong completeness). *Every consistent set of formulas* T *is* $\mathsf{P\Lambda}_{\rightarrow}^{\mathsf{Meas}}$*-satisfiable.*

Proof. We construct $\mathsf{P\Lambda}_{\rightarrow}^{\mathsf{Meas}}$-model $\mathcal{M}_{T^\star}$ and show that for every $\mathfrak{A} \in \mathsf{For}_{\mathsf{P\Lambda}_{\rightarrow}}$, $\mathcal{M}_{T^\star} \models \mathfrak{A}$ iff $\mathfrak{A} \in T^\star$. We use the induction on the complexity of the formula.

- 1) $\mathfrak{A}$ is a lambda statement, $\mathfrak{A} = M : \sigma$. If $\mathfrak{A} \in \mathsf{Cn}_{\mathsf{B}}(T)$, then by definition of $\mathcal{M}_{T^\star}$ we have $\mathcal{M}_{T^\star} \models \mathfrak{A}$. Conversely, suppose $\mathcal{M}_{T^\star} \models \mathfrak{A}$, that is, for all worlds $w \in \mathcal{M}_{T^\star}$, $w \models M : \sigma$, i.e., $[\![M]\!]_\rho^w \in [\![\sigma]\!]_\xi^w$. Let $\mathfrak{B}$ be the set of all basic statements that are in $\mathsf{Cn}_{\mathsf{B}}(T)$, i.e. $\mathfrak{B}$ is basis and $\mathfrak{B} \subseteq \mathsf{Cn}_{\mathsf{B}}(T)$. First, let us consider the case when there is an infinite number of variables that are not in $\mathfrak{B}$. We extend $\mathfrak{B}$ to a set, $\mathfrak{B}^+$, of statements in which each type is assigned to an infinite number of variables and no variable is subject of more than one statement and no variables in $\mathfrak{B}^+ \setminus \mathfrak{B}$ occurs in M (construction of a set $\mathfrak{B}^+$ can be found in [13]). Since no variable from $\mathfrak{B}^+ \setminus \mathfrak{B}$ appears in the set $\mathsf{Cn}_{\mathsf{B}}(T)$, there is a world, w_0, in which only consequences of $\mathfrak{B}^+$ hold. Let us show that $\xi_{w_0}(a) = \{[N] \mid \mathfrak{B}^+ \vdash N : a\}$ holds, i.e.,
  $\{[N] \mid w_0 \models N : a\} = \{[N] \mid \mathfrak{B}^+ \vdash N : a\}$.
  ($\subseteq$) Suppose that $[N] \notin \{[N] \mid \mathfrak{B}^+ \vdash N : a\}$, that is $\mathfrak{B}^+ \nvdash N : a$. Using Theorem 2, we obtain $\mathfrak{B}^+ \not\models N : a$. Hence, $w_0 \not\models N : a$ and therefore $[N] \notin \{[N] \mid w_0 \models N : a\}$.
  ($\supseteq$) If $[N] \in \{[N] \mid \mathfrak{B}^+ \vdash N : a\}$, we have that $\mathfrak{B}^+ \vdash N : a$. Now, by Theorem 1, we have $\mathfrak{B}^+ \models N : a$. Since, $w_0 \models \mathfrak{B}^+$, we obtain $w_0 \models N : a$ and $[N] \in \{[N] \mid w_0 \models N : a\}$.
  Furthermore, $[N] \in [\![\sigma]\!]_\xi^{w_0} \Leftrightarrow \mathfrak{B}^+ \vdash N : \sigma$ (the proof can be found in [13]). Since $M : \sigma$ holds in every world, whence in w_0 as well, we obtain $\mathfrak{B}^+ \vdash M : \sigma$. Now, the fact that M does not contain any variable from $\mathfrak{B}^+ \setminus \mathfrak{B}$, gives us that $\mathfrak{B} \vdash M : \sigma$, and so $\mathsf{Cn}_{\mathsf{B}}(T) \vdash M : \sigma$, which means that $M : \sigma \in T^\star$.
  The case when there is a finite number of variables that are not in $\mathfrak{B}$ can be proved using the same idea as in [13].
  2) $\mathfrak{A}$ is a Boolean combination of lambda statements. If $\mathfrak{A} \in \mathsf{Cn}_{\mathsf{B}}(T)$, then, again, by definition of $\mathcal{M}_{T^\star}$ we have $\mathcal{M}_{T^\star} \models \mathfrak{A}$. Conversely, let $\mathcal{M}_{T^\star} \models \mathfrak{A}$. The goal is to show that $\mathfrak{A} \in T^\star$, i.e. it is enough to show that $T \vdash \mathfrak{A}$. The Axiom 1 and Modus Ponens give us that $\mathfrak{A}$ can be proved from T because of the completeness of classical propositional calculus.

- Next, consider the case $\mathfrak{A} = P_{\geq s}\alpha$. If $P_{\geq s}\alpha \in T^\star$, then $\sup\{r \mid P_{\geq r}\alpha \in T^\star\} = \mu([\alpha]) \geq s$, and so $M_{T^\star} \models P_{\geq s}\alpha$. Conversely, suppose that $M_{T^\star} \models P_{\geq s}\alpha$, i.e. $\sup\{r \mid P_{\geq r}\alpha \in T^\star\} \geq s$. If $\mu([\alpha]) > s$, then by the properties of supremum and monotonicity of μ, we have $P_{\geq s}\alpha \in T^\star$. If $\mu([\alpha]) = s$, then, from Lemma 2, we have that $P_{\geq s}\alpha \in T^\star$.
- Further, let $\mathfrak{A} = \neg\psi \in \mathsf{For}_\mathsf{P}$. Then $\mathcal{M}_{T^\star} \models \neg\psi$ iff it is not the case that $\mathcal{M}_{T^\star} \models \psi$ iff $\psi \notin T^\star$ iff $\neg\psi \in T^\star$.
- Finally, let $\mathfrak{A} = \phi \wedge \psi \in \mathsf{For}_\mathsf{P}$. Then, $\mathcal{M}_{T^\star} \models \phi \wedge \psi$ iff $\mathcal{M}_{T^\star} \models \phi$ and $\mathcal{M}_{T^\star} \models \psi$ iff $\phi, \psi \in T^\star$ iff $\phi \wedge \psi \in T^\star$. $\square$

6 Conclusion

In this paper, we introduced the logic $\mathsf{P}\Lambda_\rightarrow$ for reasoning about probabilities of simply typed lambda terms. The language of this logic is obtained by adding the operators for probabilities and Boolean connectives to simple type assignment. An axiomatization for the logic is proposed and proved strongly complete. Since this logic is not compact, the axiomatization contains one infinitary rule of inference.

As a topic for a further research, we will work towards simplification of the semantics in order to achieve compactness using finite sets of probability values for those logics. Another goal is to provide finitary axiomatizations for those logics. Also, for a further research we want to consider a case where Axiom 1 is replaced by an Axiom that states that all *intuitionistic* propositional tautologies hold, thus to work in an intuitionistic setting. Furthermore, we want to develop a first order extension of the logic $\mathsf{P}\Lambda_\rightarrow$. Note that such a logic would extend classical first order logic, so the set of all valid formulas is not recursively enumerable [1] and no complete finitary axiomatization is possible in that undecidable framework.

Another line of research is to develop probabilistic reasoning in other type disciplines such as polymorphic, intersection and higher-order types.

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Appendix Proofs

Proof of Theorem 3. Suppose that $T \cup \{\phi\} \vdash \psi$. We use transfinite induction on the length of a proof.

If the length of a proof is equal to 1, then ψ is either an axiom or $\psi \in T \cup \{\phi\}$.

a) If ψ is an axiom:
  - $T \vdash \psi$ Ax
  - $T \vdash \psi \Rightarrow (\phi \Rightarrow \psi)$ Ax
  - $T \vdash \phi \Rightarrow \psi$ MP,

b) If $\psi \in T$:
  - $T \vdash \psi$ Hyp
  - $T \vdash \psi \Rightarrow (\phi \Rightarrow \psi)$ Ax
  - $T \vdash \phi \Rightarrow \psi$ MP,

c) If $\psi \in \{\phi\}$:
  - $T \vdash \phi \Rightarrow \phi$ Ax.

Now, suppose that the length of a proof is $k > 1$. Formula ψ can belong to the set $T \cup \{\phi\}$, but then the proof is the same as above. Therefore, suppose that the formula ψ is obtained by an application of some inference rule from the Inference Rules II.
First, if ψ is obtained by an application of Rule II.1 from $T, \phi \vdash \psi_1$ and $T, \phi \vdash \psi_1 \Rightarrow \psi$:

- $T \vdash \phi \Rightarrow \psi_1$ Ind. Hyp.
- $T \vdash \phi \Rightarrow (\psi_1 \Rightarrow \psi)$ Ind. Hyp.
- $T \vdash (\phi \Rightarrow (\psi_1 \Rightarrow \psi)) \Rightarrow ((\phi \Rightarrow \psi_1) \Rightarrow (\phi \Rightarrow \psi))$ Taut.
- $T \vdash (\phi \Rightarrow \psi_1) \Rightarrow (\phi \Rightarrow \psi)$ MP
- $T \vdash \phi \Rightarrow \psi$ MP

Next, let us consider the case $\psi = P_{\geq 1}\alpha$ is obtained from $T \cup \{\phi\}$ by an application of Rule II.2. In that case:

- $T, \phi \vdash \alpha$,
- $T, \phi \vdash P_{\geq 1}\alpha$ by IR II.2.

However, since $\alpha \in \mathsf{For_B}$ and $\phi \in \mathsf{For_P}$ (otherwise, $\phi \Rightarrow P_{\geq 1}\alpha$ would not make sense), ϕ cannot affect the proof of α from $T \cup \{\phi\}$, and we have:

(1) $T \vdash \alpha$ Hyp.
(2) $T \vdash P_{\geq 1}\alpha$ IR II.2
(3) $T \vdash P_{\geq 1}\alpha \Rightarrow (\phi \Rightarrow P_{\geq 1}\alpha)$ Taut.
(4) $T \vdash \phi \Rightarrow P_{\geq 1}\alpha$ MP.

Finally, let us consider the case $\psi = \psi_1 \Rightarrow P_{\geq s}\alpha$ is obtained from $T \cup \{\phi\}$ by an application of Rule II.3. Then:

(1) $T, \phi \vdash \psi_1 \Rightarrow P_{\geq s - \frac{1}{k}}\alpha$, for all $k \geq \frac{1}{s}$ Hyp.

(2) $T \vdash \phi \Rightarrow (\psi_1 \Rightarrow P_{\geq s-\frac{1}{k}}\alpha)$ Ind.Hyp.
(3) $T \vdash (\phi \wedge \psi_1) \Rightarrow P_{\geq s-\frac{1}{k}}\alpha$ Taut.
(4) $T \vdash (\phi \wedge \psi_1) \Rightarrow P_{\geq s}\alpha$ IR II.3
(5) $T \vdash \phi \Rightarrow \psi$ Taut. □

Proof of Theorem 4. Our goal is to show that every instance of an axiom scheme holds in every model and that the inference rules preserve the validity. The Axiom 1 holds in every model because of the completeness of classical propositional logic.
By the Definition of the finitely additive probability measure we have that $\mu([\alpha]) \geq 0$ for all $\alpha \in \mathsf{For_B}$. Hence, $\mathcal{M} \models P_{\geq 0}\alpha$, for every model $\mathcal{M}$ and the Axiom 2 is valid.
Let us consider the Axiom 3. Suppose that $P_{\leq r}\alpha$ holds in model $\mathcal{M} = \langle W, \rho, \xi, H, \mu\rangle$ and $s > r$. It means that $\mu([\alpha]) \leq r$. Since $s > r$, we obtain $\mu([\alpha]) < s$, that is $\mathcal{M} \models P_{<s}\alpha$.
Similarly, for the Axiom 4, suppose that $\mathcal{M} \models P_{<s}\alpha$. Then, we have $\mu([\alpha]) < s$, that implies $\mu([\alpha]) \leq s$. Thus, $\mathcal{M} \models P_{\leq s}\alpha$.
Next, let us consider Axiom 5. Suppose that in a model $\mathcal{M} = \langle W, \rho, \xi, H, \mu\rangle$,

$$P_{\geq r}\alpha, P_{\geq s}\beta \text{ and } P_{\geq 1}\neg(\alpha \vee \beta)$$

hold. Then, $\mu([\alpha]) \geq r$, $\mu([\beta]) \geq s$ and $[\alpha]$ and $[\beta]$ are disjoint sets. Since μ is a finitely additive measure, we have that

$$\mu([\alpha] \cup [\beta]) = \mu([\alpha \vee \beta]) = \mu([\alpha]) + \mu([\beta]).$$

Thus, $\mathcal{M} \models P_{\geq min\{1,r+s\}}(\alpha \vee \beta)$, so Axiom 5 holds in the model $\mathcal{M}$.
Now, let us consider the Axiom 6. Suppose that $P_{\leq r}\alpha$, $P_{<s}\beta$ hold in a model $\mathcal{M} = \langle W, \rho, \xi, H, \mu\rangle$. Then, $\mu([\alpha]) \leq r$ and $\mu([\beta]) < s$. From

$$[\alpha] = ([\alpha] \cap (W \setminus [\beta])) \cup [\alpha \wedge \beta],$$

follows that

$$\mu([\alpha]) \geq \mu([\alpha] \cap (W \setminus [\beta])).$$

Since $[\alpha \vee \beta] = ([\alpha] \cap (W \setminus [\beta])) \cup [\beta]$, we have that

$$\mu([\alpha \vee \beta]) \leq \mu([\alpha]) + \mu([\beta]) < r + s.$$

Therefore, $\mathcal{M} \models P_{<r+s}(\alpha \vee \beta)$.
Finally, for the Axiom 7, suppose that $P_{\geq 1}(\alpha \Rightarrow \beta)$ holds in a model $\mathcal{M} = \langle W, \rho, \xi, H, \mu\rangle$. Then, the set of all worlds in which α holds, but β does not hold has the measure 0, i.e., $\mu([\alpha] \cap (W \setminus [\alpha \Rightarrow \beta])) = 0$. From

$$[\alpha] = ([\alpha] \cap (W \setminus [\alpha \Rightarrow \beta])) \cup ([\alpha] \cap [\alpha \Rightarrow \beta])$$

follows that $\mu([\alpha]) = \mu([\alpha] \cap [\alpha \Rightarrow \beta])$ and, since $[\alpha] \cap [\alpha \Rightarrow \beta] \subseteq [\beta]$, we have that $\mu([\alpha]) \leq \mu([\beta])$. Thus, $\mathcal{M} \models P_{\geq s}\alpha \Rightarrow P_{\geq s}\beta$ and Axiom 7 holds in $\mathcal{M}$.

The proof that Inference Rules I are sound can be found in [13].

Inference Rules II:

Rule II.1 is validity-preserving for the same reason as in classical logic.

Rule II.2: suppose that α holds in $\mathcal{M} = \langle W, \rho, \xi, H, \mu \rangle$, then $[\alpha] = W$, and therefore $\mu([\alpha]) = 1$, so $\mathcal{M} \models P_{\geq 1}\alpha$.

Rule II.3: Suppose that $\mathcal{M} \models \phi \Rightarrow P_{\geq s-\frac{1}{k}}\alpha$ whenever $k \geq \frac{1}{s}$. If $\mathcal{M} \not\models \phi$, then obviously $\mathcal{M} \models \phi \Rightarrow P_{\geq s}\alpha$. Otherwise $\mathcal{M} \models P_{\geq s-\frac{1}{k}}\alpha$ for every $k \geq \frac{1}{s}$, so $\mathcal{M} \models P_{\geq s}\alpha$ because of the Archimedean properties of the set of reals (Footnote 1 on page 5). □

Proof of Lemma 1.

(1) If $T \cup \{\phi\} \vdash \bot$, and $T \cup \{\neg\phi\} \vdash \bot$, then by Deduction theorem we have $T \vdash \neg\phi$ and $T \vdash \phi$. Contradiction.

(2) Suppose that for all $n > \frac{1}{s}$:

$$T, \phi \Rightarrow \neg P_{\geq s-\frac{1}{n}}\alpha \vdash \bot.$$

Therefore, by Deduction theorem and propositional reasoning, we have

$$T \vdash \phi \Rightarrow P_{\geq s-\frac{1}{n}}\alpha,$$

and by an application of Rule II.3 we obtain $T \vdash \phi \Rightarrow P_{\geq s}\alpha$. Contradiction with the fact that $\neg(\phi \Rightarrow P_{\geq s}\alpha) \in T$. □

Proof of Lemma 2.

(1) Consequence of Definition 7.4.

(2) Let $t = sup\{s \mid P_{\geq s}\alpha \in T\} \in \mathsf{S}$. By the monotonicity of a measure, for each $s \in S$, $s < t$, $T \vdash P_{\geq s}\alpha$. Using Inference rule 3, we obtain

$$T \vdash P_{\geq t}\alpha.$$

T is a maximal consistent set of formulas, so, from (1), we have that

$$P_{\geq t}\alpha \in T.$$

□

Proof of Lemma 3.

(1) The prove that H is an algebra is straightforward using that $W = [\alpha \vee \neg\alpha]$, $[\alpha]^C = [\neg\alpha]$ and $[\alpha] \cup [\beta] = [\alpha \vee \beta]$.

(2) It suffices to prove that $[\alpha] \subset [\beta]$ implies $\mu([\alpha]) \leq \mu([\beta])$. According to the completeness of the propositional logic, we have that $[\alpha] \subset [\beta]$ means that $\alpha \Rightarrow \beta \in \mathsf{Cn}_{\mathsf{B}}(T)$, and then also $P_{\geq 1}(\alpha \Rightarrow \beta) \in T^\star$. By axiom 7, we obtain that for each $s \in S$, $P_{\geq s}\alpha \Rightarrow P_{\geq s}\beta \in T^\star$, so $\mu([\alpha]) \leq \mu([\beta])$.

(3) $P_{\geq 0}\alpha$ is an axiom, so $\mu([\alpha]) \geq 0$.

(4) For any $\alpha \in T$, we have that $\alpha \vee \neg\alpha \in \mathsf{Cn_B}(T)$ and $P_{\geq 1}(\alpha \vee \neg\alpha) \in T^\star$, therefore, we obtain that $W = [\alpha \vee \neg\alpha]$ and $\mu(W) = 1$. Since $P_{\geq 1}(\alpha \vee \neg\alpha) = P_{\geq 1-0}(\alpha \vee \neg\alpha) = P_{\leq 0}\neg(\alpha \vee \neg\alpha) = P_{\leq 0}(\neg\alpha \wedge \alpha) = \neg P_{>0}(\neg\alpha \wedge \alpha)$,
using that $P_{\geq t}\alpha \Rightarrow P_{>s}\alpha$, for $t > s$, we obtain that

$$sup\{s \mid P_{\geq s}(\neg\alpha \wedge \alpha) \in T^\star\} = 0,$$

and $\mu(\emptyset) = 0$.

(5) Let $\mu([\alpha]) = \sup\{s \mid P_{\geq s}\alpha \in T^\star\} = r$. If $r = 1$, then from Lemma **??** we obtain $P_{\geq 1}\alpha \in T^\star$. Therefore, $\neg P_{>0}\neg\alpha \in T^\star$. Again, using the fact that $P_{\geq t}\alpha \Rightarrow P_{>s}\alpha$, for $t > s$, we obtain that $\mu([\neg\alpha]) = 0$. Now, suppose that $r < 1$. Then, for each rational number $r' \in (r, 1]$, $\neg P_{\geq r'}\alpha = P_{<r'}\alpha \in T^\star$. By Axiom 4, we obtain that $P_{\leq r'}\alpha, P_{\geq 1-r'}\neg\alpha \in T^\star$. If there is some rational number $r'' \in [0, r)$ such that $P_{\geq 1-r''}\neg\alpha \in T^\star$, then $\neg P_{>r''}\alpha \in T^\star$, contradiction. Thus,

$$\sup\{s \mid P_{\geq s}\neg\alpha \in T^\star\} = 1 - \sup\{s \mid P_{\geq s}\alpha \in T^\star\},$$

i.e., $\mu([\alpha]) = 1 - \mu([\neg\alpha])$.

(6) Let $[\alpha] \cap [\beta] = \emptyset$, and let $\mu([\alpha]) = r$ and $\mu([\beta]) = s$. From the fact that $[\beta] \subset [\neg\alpha]$, using steps (2) and (5), we obtain that $r + s \leq r + (1 - r) = 1$. Suppose that both $r > 0$ and $s > 0$. Using properties of the supremum, for every rational number $r' \in [0, r)$, and for every rational number $s' \in [0, s)$, we have that $P_{\geq r'}\alpha, P_{\geq s'}\beta \in T^\star$. By the Axiom 5, we know that $P_{\geq r'+s'}(\alpha \vee \beta) \in T^\star$. Therefore, $r + s \leq t_0 = \sup\{t \mid P_{\geq t}(\alpha \vee \beta \in T^\star)\}$. In the case that $r + s = 1$, the statement holds obviously, so suppose that $r + s < 1$. If $r + s < t_0$, then for every rational number $t' \in (r + s, t_0)$ we have $P_{\geq t'}(\alpha \vee \beta) \in T^\star$. There exists rational numbers $r'' > r$ adn $s'' > s$, such that:

$$\neg P_{\geq r''}\alpha, P_{<r''}\alpha \in T^\star \quad , \neg P_{\geq s''}\alpha, P_{<s''}\alpha \in T^\star,$$

and

$$r'' + s'' = t' \leq 1.$$

By Axiom 4, we obtain $P_{\leq r''} \in T^\star$. Using Axiom 6, we get

$$P_{\leq r''+s''}(\alpha \vee \beta) \in T^\star, \quad \neg P_{\geq r''+s''}(\alpha \vee \beta) \in T^\star,$$

and

$$\neg P_{\geq t'}(\alpha \vee \beta) \in T^\star.$$

Contradiction. Hence, $r + s = t_0$ and we obtain that $\mu([\alpha] \cup [\beta]) = \mu([\alpha]) + \mu([\beta])$. Finally, if we suppose that $r = 0$ or $s = 0$, we can reason as above, where $r' = 0$ or $s' = 0$.

□