KARAKTERISTIČNI KORENI I VEKTORI MATRICE

1. Data je matrica $A = \begin{bmatrix} 0 & -1 & 0 \\ -2 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix}$.

   (a) Naći karaktristične korene i vektore matrice A.

   (b) Izračunati A^{-1} (ako postoji).

2. Data je matrica $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

   (a) Naći karaktristične korene i vektore matrice A.

   (b) Izračunati $f(A) = 2A^4 - 3A^3 + A^2 + 2A - I$.

3. Naći karaktristične korene i vektore matrice $A = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}$.

Rešenja:

(1) $A = \begin{bmatrix} 0 & -1 & 0 \\ -2 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix}$

$+\ \cdot(\lambda)$

(a) $p(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda & 1 & 0 \\ 2 & \lambda-1 & -2 \\ -2 & 0 & \lambda+1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ -\lambda^2+\lambda+2 & \lambda-1 & -2 \\ -2 & 0 & \lambda+1 \end{vmatrix} = -\begin{vmatrix} -(\lambda-2)(\lambda+1) & -2 \\ -2 & \lambda+1 \end{vmatrix}$

$= -\left(-(\lambda-2)(\lambda+1)^2 - 4\right) = (\lambda-2)(\lambda^2+2\lambda+1) + 4$

$= \lambda^3 + \cancel{2\lambda^2} + \lambda - \cancel{2\lambda^2} - 4\lambda - 2 + 4 = \lambda^3 - 3\lambda + 2$

$= (\lambda-1)^2(\lambda+2)$

λ^3	λ^2	λ	λ^0		r
1	0	-3	2		
	1	1	-2	1	0
		1	2	1	0

- karakteristični koreni: $\lambda_{1,2} = 1$, $\lambda_3 = -2$

$\boxed{\lambda = 1}$

$Ax = \lambda x \iff (A - I)x = 0 \iff \begin{bmatrix} -1 & -1 & 0 \\ -2 & 0 & 2 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{array}{l} -x_1 - x_2 = 0 \\ -2x_1 + 2x_3 = 0 \\ 2x_1 - 2x_3 = 0 \end{array}$

$\iff \begin{array}{l} -x_1 - x_2 = 0 \\ -x_1 + x_3 = 0 \\ 0 = 0 \end{array} \iff \begin{array}{l} x_2 = -x_1 \\ x_3 = x_1 \end{array} \iff x = \begin{bmatrix} \alpha \\ -\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ \alpha \in \mathbb{R}$

$\boxed{\lambda = -2}$

$Ax = \lambda x \iff (A + 2I)x = 0 \iff \begin{bmatrix} 2 & -1 & 0 \\ -2 & 3 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{array}{l} 2x_1 - x_2 = 0 \\ -2x_1 + 3x_2 + 2x_3 = 0 \\ 2x_1 + x_3 = 0 \end{array}$ $+$ $-$

$\iff \begin{array}{l} 2x_1 - x_2 = 0 \\ 2x_2 + 2x_3 = 0 \\ x_2 + x_3 = 0 \ /\cdot(-2) \end{array}$ $+$ $\iff \begin{array}{l} 2x_1 - x_2 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{array} \iff \begin{array}{l} x_1 = \frac{1}{2}x_2 \\ x_3 = -x_2 \end{array}$

$\implies x = \begin{bmatrix} \frac{1}{2}\alpha \\ \alpha \\ -\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1/2 \\ 1 \\ -1 \end{bmatrix}, \ \alpha \in \mathbb{R}$

(b) $\boxed{\text{k-H t.}} \Rightarrow p(A)=0 \Leftrightarrow A^3-3A+2I=0 \;/\cdot A^{-1}$

$$\Leftrightarrow A^2-3I+2A^{-1}=0$$

$$\Leftrightarrow A^{-1}=\frac{1}{2}(3I-A^2)$$

$$A^2=\begin{bmatrix}0&-1&0\\-2&1&2\\2&0&-1\end{bmatrix}\cdot\begin{bmatrix}0&-1&0\\-2&1&2\\2&0&-1\end{bmatrix}=\begin{bmatrix}2&-1&-2\\2&3&0\\-2&-2&1\end{bmatrix}$$

$$A^{-1}=\frac{1}{2}\left(\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}-\begin{bmatrix}2&-1&-2\\2&3&0\\-2&-2&1\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}1&1&2\\-2&0&0\\2&2&2\end{bmatrix}=\begin{bmatrix}1/2&1/2&1\\-1&0&0\\1&1&1\end{bmatrix}$$

2) $A=\begin{bmatrix}1&1&1\\0&1&0\\0&1&0\end{bmatrix}$

(a) $p(\lambda)=\begin{vmatrix}\lambda-1&-1&-1\\0&\lambda-1&0\\0&-1&\lambda\end{vmatrix}=(\lambda-1)\begin{vmatrix}\lambda-1&-1\\0&\lambda\end{vmatrix}=\lambda(\lambda-1)^2=\lambda^3-2\lambda^2+\lambda$

- karakteristični koreni: $\lambda_1=0$, $\lambda_{2,3}=1$

$\boxed{\lambda=0}$

$Ax=\lambda x \Leftrightarrow Ax=0 \Leftrightarrow \begin{bmatrix}1&1&1\\0&1&0\\0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Leftrightarrow \begin{matrix}x_1+x_2+x_3=0\\x_2=0\\x_2=0\end{matrix}$

$\Leftrightarrow \begin{matrix}x_3=-x_1\\x_2=0\end{matrix} \Leftrightarrow x=\begin{bmatrix}\alpha\\0\\-\alpha\end{bmatrix}=\alpha\begin{bmatrix}1\\0\\-1\end{bmatrix},\ \alpha\in\mathbb{R}$

$\boxed{\lambda=1}$

$Ax=\lambda x \Leftrightarrow (A-I)x=0 \Leftrightarrow \begin{bmatrix}0&1&1\\0&0&0\\0&1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Leftrightarrow \begin{matrix}x_2+x_3=0\\0=0\\x_2-x_3=0\end{matrix}$

$\Leftrightarrow \begin{matrix}x_2=x_3=0\\x_1\in\mathbb{R}\end{matrix} \Leftrightarrow x=\begin{bmatrix}\alpha\\0\\0\end{bmatrix}=\alpha\begin{bmatrix}1\\0\\0\end{bmatrix},\ \alpha\in\mathbb{R}$

(b) $\boxed{\text{k-H t.}} \Rightarrow p(A)=0 \Leftrightarrow A^3-2A^2+A=0$

$$\begin{aligned}f(A)&=2A^4-3A^3+A^2+2A-I\\&=2A(A^3-2A^2+A)+A^3-A^2+2A-I\\&=(A^3-2A^2+A)+A^2+A-I\\&=A^2+A-I\end{aligned}$$

$$A^2=\begin{bmatrix}1&1&1\\0&1&0\\0&1&0\end{bmatrix}\cdot\begin{bmatrix}1&1&1\\0&1&0\\0&1&0\end{bmatrix}=\begin{bmatrix}1&3&1\\0&1&0\\0&1&0\end{bmatrix}$$

$$f(A)=\begin{bmatrix}1&3&1\\0&1&0\\0&1&0\end{bmatrix}+\begin{bmatrix}1&1&1\\0&1&0\\0&1&0\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&4&2\\0&1&0\\0&2&-1\end{bmatrix}$$

③ $A = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}$

$$p(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-a & -b & -b \\ -b & \lambda-a & -b \\ -b & -b & \lambda-a \end{vmatrix} = \begin{vmatrix} \lambda-a-2b & -b & -b \\ \lambda-a-2b & \lambda-a & -b \\ \lambda-a-2b & -b & \lambda-a \end{vmatrix} \begin{matrix} \cdot(-1) \\ + \\ + \end{matrix}$$

$$= \begin{vmatrix} \lambda-a-2b & -b & -b \\ 0 & \lambda-a+b & 0 \\ 0 & 0 & \lambda-a+b \end{vmatrix} = (\lambda-a-2b)(\lambda-a+b)^2$$

• karakteristični koreni: $\lambda_1 = a+2b$, $\lambda_{2,3} = a-b$

$\boxed{\lambda = a+2b}$

$$Ax = \lambda x \iff (A-(a+2b)I)x = 0 \iff \begin{bmatrix} -2b & b & b \\ b & -2b & b \\ b & b & -2b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff b \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- $b = 0 \Rightarrow x \in \mathbb{R}^3$
- $b \neq 0$

$$\begin{aligned} -2x_1 + x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 + x_2 - 2x_3 &= 0 \;/\cdot(-1)\;/\cdot 2 \end{aligned} \iff \begin{aligned} x_1 + x_2 - 2x_3 &= 0 \\ -3x_2 + 3x_3 &= 0 \\ 3x_2 - 3x_3 &= 0 \end{aligned} \iff \begin{aligned} x_1 + x_2 - 2x_3 &= 0 \\ -x_2 + x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\iff x_1 = x_2 = x_3 \iff x = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \alpha \in \mathbb{R}$$

$\boxed{\lambda = a-b}$

$$Ax = \lambda x \iff (A-(a-b)I)x = 0 \iff \begin{bmatrix} b & b & b \\ b & b & b \\ b & b & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff b \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- $b = 0 \Rightarrow x \in \mathbb{R}^3$
- $b \neq 0$

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \;/\cdot(-1) \\ x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned} \iff \begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned} \iff x_1 = -x_2 - x_3$$

$$\iff x = \begin{bmatrix} -\alpha-\beta \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -\alpha \\ \alpha \\ 0 \end{bmatrix} + \begin{bmatrix} -\beta \\ 0 \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \alpha, \beta \in \mathbb{R}$$

LINEARNE TRANSFORMACIJE

1. Za date funkcije ispitati da li su linearne transformacije, i za one koje jesu naći matricu linearne transformacije i odrediti njen rang:

   (a) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x^2 - y, x + y)$;

   (b) $g : \mathbb{R}^2 \to \mathbb{R}^3$, $g(x,y) = (2x - y, 3x, y + 1)$;

   (c) $h : \mathbb{R}^3 \to \mathbb{R}^2$, $h(x,y,z) = (x - y + 2z, -x + 3y + z)$.

2. Za sledeće funkcije diskutovati po realnim parametrima kada su linearne transformacije, i u slučaju kada jesu naći njihove matrice i odrediti rang.

   (a) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x,y,z) = (ax + y^b, bx - z)$;

   (b) $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = ax + bxy + cy$;

   (c) $h : \mathbb{R}^3 \to \mathbb{R}^2$, $h(x,y,z) = (a^x + yz^b, ax + by + cz)$.

3. Neka su linearne transformacije $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ definisane sa $f(x,y) = (2x - y, x + 3y)$ i $g(x,y) = (-x + y, 3x - 2y)$

   (a) Odrediti kompoziciju $f \circ g$.

   (b) Napisati matrice M_f i M_g za transformacije f i g.

   (c) Naći linearnu transformaciju h koja odgovara matrici $M_f \cdot M_g$ i uporediti je sa $f \circ g$.

   (d) Odrediti f^{-1} i g^{-1} ako postoje.

4. Za linearnu transformaciju $f : \mathbb{R}^2 \to \mathbb{R}^2$ važi $f(1,-1) = (-3,6)$ i $f(-2,1) = (2,-4)$. Izračunati $f(x,y)$ i odgovarajuću matricu M linearne transformacije f, i odrediti njen rang. Da li postoji f^{-1}?

5. Linearna transformacija $f : \mathbb{R}^3 \to \mathbb{R}^2$ data je sa $f(5,-1,3) = (1,2)$, $f(-4,1,-2) = (-3,0)$ i $f(6,-1,3) = (4,-1)$. Izračunati $f(3,1,1)$.

Rešenja:

1 (a) $f(x,y) = (x^2 - y, x + y)$

- $f((a,b)+(c,d)) \overset{?}{=} f(a,b) + f(c,d)$

$f((a,b)+(c,d)) = f(a+c, b+d) = ((a+c)^2 - (b+d), (a+c)+(b+d))$

$= (a^2 + 2ac + c^2 - b - d, a + b + c + d)$

$f(a,b) + f(c,d) = (a^2 - b, a + b) + (c^2 - d, c + d)$

$= (a^2 + c^2 - b - d, a + b + c + d)$

$\Rightarrow f((a,b)+(c,d)) \neq f(a,b) + f(c,d)$, ako je $a \cdot c \neq 0$

$\Rightarrow f$ nije linearna transformacija

(b) $g(x,y) = (2x-y,\ 3x,\ y+1)$

- $g((a,b)+(c,d)) \overset{?}{=} g(a,b) + g(c,d)$

$g((a,b)+(c,d)) = g(a+c, b+d) = (2(a+c)-(b+d),\ 3(a+c),\ b+d+1)$

$= (2a+2c-b-d,\ 3a+3c,\ b+d+\underline{\underline{1}})$

$g(a,b)+g(c,d) = (2a-b,\ 3a,\ b+1) + (2c-d,\ 3c,\ d+1)$

$= (2a+2c-b-d,\ 3a+3c,\ b+d+\underline{\underline{2}})$

$\Rightarrow g((a,b)+(c,d)) \neq g(a,b)+g(c,d)$

$\Rightarrow$ g nije linearna transformacija

(c) $h(x,y,z) = (x-y+2z,\ -x+3y+z)$

- $h((a,b,c)+(d,e,f)) \overset{?}{=} h(a,b,c) + h(d,e,f)$

$h((a,b,c)+(d,e,f)) = h(a+d,\ b+e,\ c+f)$

$= (a+d-b-e+2c+2f,\ -a-d+3b+3e+c+f)$

$h(a,b,c)+h(d,e,f) = (a-b+2c,\ -a+3b+c) + (d-e+2f,\ -d+3e+f)$

$= (a+d-b-e+2c+2f,\ -a-d+3b+3e+c+f)$

$\Rightarrow$ važi jednakost

- $h(\alpha(a,b,c)) \overset{?}{=} \alpha h(a,b,c)$

$h(\alpha(a,b,c)) = h(\alpha a, \alpha b, \alpha c) = (\alpha a - \alpha b + 2\alpha c,\ -\alpha a + 3\alpha b + \alpha c)$

$\alpha h(a,b,c) = \alpha(a-b+2c,\ -a+3b+c) = (\alpha a - \alpha b + 2\alpha c,\ -\alpha a + 3\alpha b + \alpha c)$

$\Rightarrow$ važi jednakost

Dakle, h jeste linearna transformacija

(2) (a) $f(x,y,z) = (ax + \underset{\substack{\downarrow \\ b=1}}{y^b},\ bx - z) = (ax+y,\ x-z)$, $a \in \mathbb{R}$

$$M_f = \begin{bmatrix} a & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \rightarrow \text{rang}(M_f) = 2$$

(b) $g(x,y) = \underset{a\in\mathbb{R}}{ax} + \underset{\substack{\downarrow \\ b=0}}{bxy} + \underset{c\in\mathbb{R}}{cy} = ax + cy$

$$M_g = [a, c] \rightarrow \text{rang}(M_g) = \begin{cases} 0, & a=c=0 \\ 1, & \text{inače} \end{cases}$$

(c) $h(x,y,z) = (\underset{\substack{\downarrow \\ a=0}}{a^x} + \underset{\substack{\downarrow \\ b=0}}{yz^b},\ ax+by+\underset{c\in\mathbb{R}}{cz}) = (y,\ cz)$

$$M_h = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix} \rightarrow \text{rang}(M_h) = \begin{cases} 1, & c=0 \\ 2, & c\neq 0 \end{cases}$$

③ $f(x,y)=(2x-y,\ x+3y)$

$g(x,y)=(-x+y,\ 3x-2y)$

(a) $(f\circ g)(x,y)=f(g(x,y)) = f(-x+y,\ 3x-2y)$

$= (2(-x+y)-(3x-2y),\ -x+y+3(3x-2y))$

$= (-5x+4y,\ 8x-5y)$

(b) $M_f=\begin{bmatrix}2&-1\\1&3\end{bmatrix}$ $\qquad M_g=\begin{bmatrix}-1&1\\3&-2\end{bmatrix}$

(c) $M_f\cdot M_g=\begin{bmatrix}2&-1\\1&3\end{bmatrix}\cdot\begin{bmatrix}-1&1\\3&-2\end{bmatrix}=\begin{bmatrix}-5&4\\8&-5\end{bmatrix}=M_h \Rightarrow h(x,y)=(-5x+4y,\ 8x-5y)$

$\Downarrow$

$h(x,y)=(f\circ g)(x,y)$

(d) Odredićemo najpre M_f^{-1} i M_g^{-1}.

$|M_f|=\begin{vmatrix}2&-1\\1&3\end{vmatrix}=7 \Rightarrow M_f^{-1}=\frac{1}{7}\begin{bmatrix}3&1\\-1&2\end{bmatrix} \Rightarrow f^{-1}(x,y)=\frac{1}{7}(3x+y,\ -x+2y)$

$|M_g|=\begin{vmatrix}-1&1\\3&-2\end{vmatrix}=-1 \Rightarrow M_g^{-1}=-\begin{bmatrix}-2&-1\\-3&-1\end{bmatrix}=\begin{bmatrix}2&1\\3&1\end{bmatrix} \Rightarrow g^{-1}(x,y)=(2x+y,\ 3x+y)$

④ $f(1,-1)=(-3,6)$

$f(-2,1)=(2,-4)$

I način:

$f(x,y)=f(x\cdot(1,0)+y\cdot(0,1))=x\cdot f(1,0)+y\cdot f(0,1) \Rightarrow$ treba odrediti slike vektora standardne baze!

$\alpha(1,-1)+\beta(-2,1)=(1,0)$

$\begin{matrix}\alpha-2\beta=1\\-\alpha+\beta=0\end{matrix} \Leftrightarrow \left.\begin{matrix}\alpha=-1\\\beta=-1\end{matrix}\right\} \Rightarrow f(1,0)=f(-(1,-1)-(-2,1))$

$= -f(1,-1)-f(-2,1)$

$= -(-3,6)-(2,-4)$

$= (1,-2)$

$\gamma(1,-1)+\delta(-2,1)=(0,1)$

$\begin{matrix}\gamma-2\delta=0\\-\gamma+\delta=1\end{matrix} \Leftrightarrow \left.\begin{matrix}\gamma=-2\\\delta=-1\end{matrix}\right\} \Rightarrow f(0,1)=f(-2(1,-1)-(-2,1))$

$= -2f(1,-1)-f(-2,1)$

$= -2(-3,6)-(2,-4)$

$= (4,-8)$

$f(x,y)=x\cdot(1,-2)+y(4,-8)$

$= (x+4y,\ -2x-8y)$

$M_f=\begin{bmatrix}1&4\\-2&-8\end{bmatrix}\sim\begin{bmatrix}1&4\\0&0\end{bmatrix} \rightarrow \mathrm{rang}(M_f)=1 \rightarrow$ ne postoji f^{-1}

II način:

$$f(1,-1)=f(e_1-e_2)=\boxed{f(e_1)-f(e_2)=(-3,6)}$$
$$f(-2,1)=f(-2e_1+e_2)=\boxed{-2f(e_1)+f(e_2)=(2,-4)}$$
$$\Leftrightarrow \begin{bmatrix}1&-1\\-2&1\end{bmatrix}\cdot\begin{bmatrix}f(e_1)\\f(e_2)\end{bmatrix}=\begin{bmatrix}(-3,6)\\(2,-4)\end{bmatrix} \quad (*)$$

$$\Leftrightarrow \begin{bmatrix}f(e_1)\\f(e_2)\end{bmatrix}=\begin{bmatrix}1&-1\\-2&1\end{bmatrix}^{-1}\cdot\begin{bmatrix}(-3,6)\\(2,-4)\end{bmatrix}=-\begin{bmatrix}1&1\\2&1\end{bmatrix}\cdot\begin{bmatrix}(-3,6)\\(2,-4)\end{bmatrix}=\begin{bmatrix}(1,-2)\\(4,-8)\end{bmatrix}$$

$$\Leftrightarrow \begin{array}{l}f(e_1)=(1,-2)\\f(e_2)=(4,-8)\end{array} \rightarrow M_f=\begin{bmatrix}1&4\\-2&-8\end{bmatrix}$$

III način:

$$A=\begin{bmatrix}1&-2\\-1&1\end{bmatrix} \qquad B=\begin{bmatrix}-3&2\\6&-4\end{bmatrix}$$

$$(*)\Leftrightarrow A^T\cdot M_f^T=B^T \Rightarrow M_f^T=(A^T)^{-1}\cdot B^T=(A^{-1})^T\cdot B^T=(B\cdot A^{-1})^T$$
$$\Rightarrow M_f=B\cdot A^{-1}$$

$$M_f=\begin{bmatrix}-3&2\\6&-4\end{bmatrix}\cdot\begin{bmatrix}-1&-2\\-1&-1\end{bmatrix}=\begin{bmatrix}1&4\\-2&-8\end{bmatrix}$$

5) $f(5,-1,3)=(1,2)$
$f(-4,1,-2)=(-3,0)$
$f(6,-1,3)=(4,-1)$

I način

$$A=\begin{bmatrix}5&-4&6\\-1&1&-1\\3&-2&3\end{bmatrix} \qquad B=\begin{bmatrix}1&-3&4\\2&0&-1\end{bmatrix}$$

$$M_f=B\cdot A^{-1}=\begin{bmatrix}1&-3&4\\2&0&-1\end{bmatrix}\cdot\begin{bmatrix}-1&0&2\\0&3&1\\1&2&-1\end{bmatrix}=\begin{bmatrix}3&-1&-5\\-3&-2&5\end{bmatrix}$$

$$[f(3,1,1)]=\begin{bmatrix}3&-1&-5\\-3&-2&5\end{bmatrix}\cdot\begin{bmatrix}3\\1\\1\end{bmatrix}=\begin{bmatrix}3\\-6\end{bmatrix} \Leftrightarrow f(3,1,1)=(3,-6)$$

II način

$$\alpha(5,-1,3)+\beta(-4,1,-2)+\gamma(6,-1,3)=(3,1,1)$$
$$\begin{array}{l}5\alpha-4\beta+6\gamma=3\\-\alpha+\beta-\gamma=1\\3\alpha-2\beta+3\gamma=1\end{array} \Leftrightarrow (\alpha,\beta,\gamma)=(-1,4,4)$$

$$\begin{aligned}f(3,1,1)&=f(-(5,-1,3)+4(-4,1,-2)+4(6,-1,3))\\&=-f(5,-1,3)+4f(-4,1,-2)+4f(6,-1,3)\\&=-(1,2)+4(-3,0)+4(4,-1)\\&=(3,-6)\end{aligned}$$