POLINOMI

1. Odrediti količnik i ostatak pri deljenju polinoma $p(x) = (x+1)(x+3)(x+4)$ polinomom $q(x) = x^2 + 1$.
2. Naći racionalne korene polinoma $p(x) = 3x^4 + 5x^3 + x^2 + 5x - 2$, i faktorisati ga nad poljem realnih i nad poljem kompleksnih brojeva.
3. Neka je $p(z) = iz^3 + z^2 + 2z - 2i$ polinom nad $\mathbb{C}$.
   (i) Pokazati da je $z_1 = i$ nula polinoma p.
   (ii) Pokazati da $\bar{z}_1$ nije nula polinoma p.
   (iii) Faktorisati polinom p nad $\mathbb{C}$.
4. Odrediti normiran polinom najmanjeg stepena
   (i) nad $\mathbb{C}$
   (ii) nad $\mathbb{C}$, sa realnim koeficijentima,

   tako da broj -1 bude dvostruki, a brojevi 2 i $1 - i$ jednostruki koreni tog polinoma.
5. Polinom $p(x) = x^5 + x^3 + 2x^2 - 12x + 8$ napisati po stepenima od $x + 1$.
6. Ostatak pri deljenju polinoma $p(x)$ polinomom $x - 1$ je 3, a polinomom $x + 2$ je -3. Naći ostatak pri deljenju polinoma $p(x)$ polinomom $q(x) = x^2 + x - 2$.
7. Naći normiran polinom $p(x)$ četvrtog stepena ako se zna da je zbir njegovih korena 2, proizvod 1, i ako pri deljenju sa $x - 2$ daje ostatak 5, a pri deljenju sa $x + 1$ daje ostatak 8.
8. Neka je $p(z) = z^3 + pz^2 + qz + r$, $p, q, r \in \mathbb{C}$, i neka su z_1, z_2, z_3 koreni polinoma p. Ako brojevima $0, z_1, z_2, z_3$ u kompleksnoj ravni odgovaraju temena paralelograma, dokazati da je $p^3 - 4pq + 8r = 0$.
9. Naći najveći zajednički delilac za polinome $p(x) = x^6 + 3x^5 - 11x^4 - 27x^3 + 10x^2 + 24x$ i $q(x) = x^3 - 2x^2 - x + 2$.
10. Da li postoje realni brojevi a i b takvi da je nad poljem realnih brojeva najveći zajednički delilac polinoma
    $$p(x) = x^5 - ax^3 + 2bx^2 + 4, \qquad q(x) = x^4 + 2x^3 - x - 2$$
    polinom $r(x) = x^2 + x - 2$?
11. Neka je dat polinom
    $$p(x) = x^5 + ax^4 + 3x^3 + bx^2 + cx.$$
    (a) Odrediti realne koeficijente a, b, c polinoma p tako da bude deljiv sa $x^2 + 1$ i $x - 1$.
    (b) Odrediti najveći zajednički delilac polinoma p i q, ako je $q(x) = x^3 - 3x - 2$.
    (c) Napisati u obliku zbira parcijalnih razlomaka racionalnu funkciju $r(x) = \frac{q(x)}{p(x)}$.
12. Rastaviti na zbir parcijalnih razlomaka racionalnu funkciju $r(x) = \dfrac{2x^4 - x^3 - 11x - 2}{x^4 + 2x^3 + 2x^2 + 2x + 1}$.
13. Faktorisati polinom $p(x) = 27x^6 - 9x^4 + 3x^2 - 1$ nad poljima $\mathbb{C}, \mathbb{R}$ i $\mathbb{Q}$.
14. Dokazati da je polinom $p(x) = x^8 + x^4 + 1$ deljiv polinomom $q(x) = x^2 + x + 1$.

Rešenja:

1. $p(x) = (x+1)(x+3)(x+4) = x^3+8x^2+19x+12$

$$\begin{array}{l}(x^3+8x^2+19x+12):(x^2+1) = x+8\\ \underline{-x^3 \qquad -x}\\ \quad 8x^2+18x+12\\ \quad \underline{-8x^2 \qquad -8}\\ \qquad 18x+4\end{array}$$

$$p(x) = q(x)\cdot \underbrace{(x+8)}_{\text{količnik}} + \underbrace{18x+4}_{\text{ostatak}}$$

2. $p(x) = 3x^4+5x^3+x^2+5x-2$

$$\left.\begin{array}{l} p \mid (-2) \Rightarrow p \in \{\pm 1, \pm 2\} \\ q \mid 3 \Rightarrow q \in \{\pm 1, \pm 3\}\end{array}\right\} \Rightarrow \frac{p}{q} \in \{\pm 1, \pm 2, \pm \tfrac{1}{3}, \pm \tfrac{2}{3}\}$$

x^4	x^3	x^2	x^1	x^0	p/q	r
3	5	1	5	-2		
	3	-1	3	-1	-2	0
		3	0	3	$\frac{1}{3}$	0

$p(x) = (x+2)(x-\frac{1}{3})(3x^2+3)$

$= (x+2)(3x-1)(x^2+1)$ faktorizacija nad $\mathbb{R}$

$p(x) = (x+2)(3x-1)(x-i)(x+i)$ faktorizacija nad $\mathbb{C}$

3. $p(z) = iz^3+z^2+2z-2i$

(i) $z_1 = i \Rightarrow p(i) = i\cdot i^3+i^2+2\cdot i-2i = 1-1+2i-2i = 0$

(ii) $\bar{z}_1 = -i \Rightarrow p(-i) = i\cdot(-i)^3+(-i)^2+2\cdot(-i)-2i = -1-1-2i-2i = -2-4i \neq 0$

(iii)

z^3	z^2	z^1	z^0		r
i	1	2	$-2i$		
	i	0	2	i	0

$p(z) = (z-i)(iz^2+2)$

$iz^2+2=0 \Leftrightarrow z^2 = 2i = 2e^{\frac{\pi}{2}i}$

$z_{2,3} = \sqrt{2}\, e^{\frac{\frac{\pi}{2}+2k\pi}{2}i}, \quad k=0,1$

$k=0 \to z_2 = \sqrt{2}\,e^{\frac{\pi}{4}i} = \sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right) = 1+i$

$k=1 \to z_3 = \sqrt{2}\,e^{\frac{5\pi}{4}i} = \sqrt{2}\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right) = -1-i$

$p(z) = i(z-i)(z-1-i)(z+1+i)$

4. (i) <u>koreni</u>:

$x_{1,2} = -1$

$x_3 = 2$

$x_4 = 1-i$

$p(x) = (x+1)^2(x-2)(x-1+i) = (x^2+2x+1)(x^2+(-3+i)x+2-2i)$

$= x^4+(-1+i)x^3-3x^2+(1-3i)x+2-2i$

(ii) <u>koreni</u>:

$x_{1,2} = -1$

$x_3 = 2$

$x_4 = 1-i$

$x_5 = 1+i$

$p(x) = (x+1)^2(x-2)(x-1+i)(x-1-i)$

$= (x^2+2x+1)(x-2)((x-1)^2+1)$

$= (x^3-3x-2)(x^2-2x+2)$

$= x^5-2x^4-x^3+4x^2-2x-4$

(5)

x^5	x^4	x^3	x^2	x^1	x^0		r
1	0	1	2	-12	8		
	1	-1	2	0	-12	-1	20
		1	-2	4	-4	-1	-8
			1	-3	7	-1	-11
				1	-4	-1	11
					1	-1	-5
							1

$$\begin{aligned}
p(x) &= (x+1)(x^4-x^3+2x^2-12)+20\\
&= (x+1)\big((x+1)(x^3-2x^2+4x-4)-8\big)+20\\
&= (x+1)\Big((x+1)\big((x+1)(x^2-3x+7)-11\big)-8\Big)+20\\
&= (x+1)\Big((x+1)\Big((x+1)\big((x+1)(x-4)+11\big)-11\Big)-8\Big)+20\\
&= (x+1)\Big((x+1)\Big((x+1)\Big((x+1)\big((x+1)\cdot 1-5\big)+11\Big)-11\Big)-8\Big)+2\\
&= 1\cdot(x+1)^5-5(x+1)^4+11(x+1)^3-11(x+1)^2-8(x+1)+20
\end{aligned}$$

(6) Prema Bezuovom stavu imamo $p(1)=3$ i $p(-2)=-3$

$p(x)=q(x)\cdot s(x)+r(x)$, $st(r)<st(q) \Rightarrow st(r)\le 1$

$p(x)=(x-1)\cdot(x+2)\cdot s(x)+ax+b$

$$\left.\begin{array}{l} p(1)=0+a+b=3 \\ p(-2)=0-2a+b=-3 \end{array}\right\} \Rightarrow a=2 \wedge b=1 \Rightarrow r(x)=2x+1$$

(7) $p(x)=x^4+ax^3+bx^2+cx+d$

Neka su x_1, x_2, x_3, x_4 koreni polinoma $p(x)$. Na osnovu Vijetovih formula sledi:

$x_1+x_2+x_3+x_4=\frac{-a}{1}=2 \Rightarrow a=-2$

$x_1\cdot x_2\cdot x_3\cdot x_4=\frac{d}{1}=1 \Rightarrow d=1$

Sada je $p(x)=x^4-2x^3+bx^2+cx+1$.

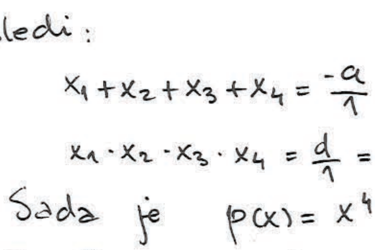
Iz Bezuovog stava i preostalih uslova zadatka sledi

$$\left.\begin{array}{l} p(2)=5 \Rightarrow 5=2^4-2\cdot 2^3+b\cdot 2^2+c\cdot 2+1=4b+2c+1 \Rightarrow 2b+c=2 \\ p(-1)=8 \Rightarrow 8=(-1)^4-2\cdot(-1)^3+b\cdot(-1)^2+c\cdot(-1)+1=b-c+4 \Rightarrow b-c=4 \end{array}\right\} \Rightarrow \begin{array}{l} b=2 \\ c=-2 \end{array}$$

Traženi polinom je $p(x)=x^4-2x^3+2x^2-2x+1$.

(8) $p(z)=z^3+pz^2+qz+r$, $p,q,r\in\mathbb{C}$

Neka su z_1, z_2, z_3 koreni od p



Kako je $Oz_1z_2z_3$ paralelogram, sledi $z_2=z_1+z_3$.

Iz Vijetovih formula imamo sledeće veze:

$z_1+z_2+z_3=-p \Rightarrow 2z_2=-p \Rightarrow z_2=-\frac{p}{2}$

$z_1\cdot z_2\cdot z_3=-r \Rightarrow -\frac{p}{2}\cdot z_1z_3=-r \Rightarrow z_1z_3=\frac{2r}{p}$

$z_1z_2+z_1z_3+z_2z_3=q$

$z_2(z_1+z_3)+z_1z_3=q$

$\frac{p^2}{4}+\frac{2r}{p}=q \quad /\cdot 4p$

$p^3+8r=4pq$

(9) $p(x) = x^6 + 3x^5 - 11x^4 - 27x^3 + 10x^2 + 24x$

$q(x) = x^3 - 2x^2 - x + 2$

I način

Faktorisaćemo polinome p i q

$p(x) = x(x^5 + 3x^4 - 11x^3 - 27x^2 + 10x + 24) = x \cdot p_1(x)$

Sve moguće racionalne nule polinoma p_1 su u skupu

$\{\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24\}$.

x^5	x^4	x^3	x^2	x^1	x^0		r
1	3	-11	-27	10	24		
	1	4	-7	-34	-24	1	0
		1	3	-10	-24	-1	0
			1	1	-12	-2	0
				1	4	3	0

$p(x) = x\boxed{(x-1)(x+1)}(x+2)(x-3)(x+4)$

$q(x) = x^3 - 2x^2 - x + 2 = x^2(x-2) - (x+2)$

$= (x^2-1)(x-2) = \boxed{(x-1)(x+1)}(x-2)$

$NZD(p, q) = x^2 - 1$

II način : Euklidov algoritam

$(x^6 + 3x^5 - 11x^4 - 27x^3 + 10x^2 + 24x) : (x^3 - 2x^2 - x + 2) = x^3 + 5x^2 - 24$

$-x^6 + 2x^5 + x^4 - 2x^3$

$5x^5 - 10x^4 - 29x^3 + 10x^2$

$-5x^5 + 10x^4 + 5x^3 - 10x^2$

$-24x^3 \qquad + 24x$

$+24x^3 - 48x^2 - 24x + 48$

$-48x^2 \qquad + 48 = -48(x^2-1)$

$(x^3 - 2x^2 - x + 2) : (x^2 - 1) = x - 2$

$-x^3 \qquad + x$

$-2x^2 \qquad + 2$

$2x^2 \qquad - 2$

0

$\Rightarrow NZD(p, q) = x^2 - 1$

(10) $p(x) = x^5 - ax^3 + 2bx^2 + 4$

$q(x) = x^4 + 2x^3 - x - 2$

$r(x) = x^2 + x - 2 = (x+2)(x-1)$

$r(x) \mid p(x) \Rightarrow (x+2) \mid p(x) \Rightarrow p(-2) = 0$

$(x-1) \mid p(x) \Rightarrow p(1) = 0$

$0 = (-2)^5 - a(-2)^3 + 2b(-2)^2 + 4$

$= -32 + 8a + 8b + 4$

$= 8a + 8b - 28$

$0 = 1^5 - a \cdot 1^3 + 2b \cdot 1^2 + 4$

$= -a + 2b + 5$

$\left.\begin{array}{l} 2a + 2b = 7 \\ a - 2b = 5 \end{array}\right\} \Rightarrow a = 4 \land b = -\frac{1}{2}$

$p(x) = x^5 - 4x^3 - x^2 + 4 = x^3(x^2-4) - (x^2-4)$

$= (x^2-4)(x^3-1) = (x-2)\boxed{(x+2)(x-1)(x^2+x+1)}$

$q(x) = x^4 + 2x^3 - x - 2 = x^3(x+2) - (x+2)$

$= (x+2)(x^3-1) = \boxed{(x+2)(x-1)(x^2+x+1)}$

$NZD(p, q) = q \neq r$

(11) $p(x) = x^5 + ax^4 + 3x^3 + bx^2 + cx$

(a) $(x^2+1) \mid p(x)$

$$
\begin{array}{l}
(x^5 + ax^4 + 3x^3 + bx^2 + cx) : (x^2+1) = x^3 + ax^2 + 2x + b - a \\
\underline{-x^5 \qquad\quad -x^3} \\
\qquad ax^4 + 2x^3 + bx^2 \\
\quad \underline{-ax^4 \qquad\quad -ax^2} \\
\qquad\qquad 2x^3 + (b-a)x^2 + cx \\
\qquad\quad \underline{-2x^3 \qquad\qquad\quad -2x} \\
\qquad\qquad\qquad (b-a)x^2 + (c-2)x \\
\qquad\qquad\quad \underline{-(b-a)x^2 \qquad\quad -(b-a)} \\
\qquad\qquad\qquad\qquad (c-2)x + a - b = 0
\end{array}
$$

$c - 2 = 0 \Rightarrow c = 2$

$a - b = 0 \Rightarrow a = b$

$p(x) = x^5 + ax^4 + 3x^3 + ax^2 + 2x$

$(x-1) \mid p(x) \Rightarrow p(1) = 0 = 1^5 + a\cdot 1^4 + 3\cdot 1^3 + a \cdot 1^2 + 2 \cdot 1 = 2a + 6$

$\Rightarrow a = b = -3$

$p(x) = x^5 - 3x^4 + 3x^3 - 3x^2 + 2x$

(b) $p(x) = (x^2+1)\cdot(x^3 - 3x^2 + 2x)$

$= (x^2+1) \cdot x\,(x^2 - 3x + 2)$

$= x\,(x^2+1)\boxed{(x-2)}(x-1)$

$q(x) = x^3 - 3x - 2 = (x+1)^2 \boxed{(x-2)}$

x^3	x^2	x	x^0		
1	0	-3	-2		
	1	-1	-2	-1	0
		1	-2	-1	0

$NZD(p, q) = x - 2$

(c) $r(x) = \dfrac{q(x)}{p(x)} = \dfrac{(x+1)^2 \cancel{(x-2)}}{x(x^2+1)\cancel{(x-2)}(x-1)} = \dfrac{x^2+2x+1}{x(x-1)(x^2+1)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+1}$

$x^2 + 2x + 1 = A(x-1)(x^2+1) + Bx(x^2+1) + (Cx+D)\,x(x-1)$

$= Ax^3 - Ax^2 + Ax - A + Bx^3 + Bx + Cx^3 - Cx^2 + Dx^2 - Dx$

$= (A+B+C)x^3 + (-A-C+D)x^2 + (A+B-D)x - A$

$$
\begin{array}{l}
A + B + C = 0 \\
-A - C + D = 1 \\
A + B - D = 2 \\
-A = 1 \Rightarrow A = -1
\end{array}
\qquad
\begin{array}{l}
B + C = 1 \\
-C + D = 0 \\
B - D = 3
\end{array}
\; +
\qquad
\begin{array}{l}
B + C = 1 \\
B - C = 3 \\
C = D
\end{array}
\qquad
\begin{array}{l}
B = 2 \\
C = D = -1
\end{array}
$$

$r(x) = -\dfrac{1}{x} + \dfrac{2}{x-1} - \dfrac{x+1}{x^2+1}$

(12) $r(x) = \dfrac{2x^4 - x^3 - 11x - 2}{x^4 + 2x^3 + 2x^2 + 2x + 1} = 2 - \dfrac{5x^3 + 4x^2 + 15x + 4}{x^4 + 2x^3 + 2x^2 + 2x + 1} = 2 - \dfrac{5x^3 + 4x^2 + 15x + 4}{(x+1)^2(x^2+1)}$

$$
\begin{array}{l}
(2x^4 - x^3 \qquad -11x - 2) : (x^4 + 2x^3 + 2x^2 + 2x + 1) = 2 \\
\underline{-2x^4 - 4x^3 - 4x^2 - 4x - 2} \\
\qquad -5x^3 - 4x^2 - 15x - 4
\end{array}
$$

$x^4 + 2x^3 + 2x^2 + 2x + 1 = x^4 + 2x^3 + x^2 + x^2 + 2x + 1$

$= x^2(x^2 + 2x + 1) + (x^2 + 2x + 1)$

$= (x+1)^2(x^2+1)$

$$r(x) = 2 - \left(\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}\right)$$

$$\begin{aligned} 5x^3+4x^2+15x+4 &= A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2 \\ &= Ax^3+Ax^2+Ax+A + Bx^2+B + Cx^3+2Cx^2+Cx + Dx^2+2Dx+D \\ &= (A+C)x^3 + (A+B+2C+D)x^2 + (A+C+2D)x + A+B+D \end{aligned}$$

$$\begin{array}{l} A + C = 5 \\ A+B+2C+D = 4 \\ A + C + 2D = 15 \\ A+B+D = 4 \end{array} \qquad \begin{array}{l} 2D = 10 \Rightarrow D = 5 \\ 2C = 0 \Rightarrow C = 0 \\ A = 5 - C = 5 \\ B = 4 - A - D = -6 \end{array} \qquad r(x) = 2 - \left(\frac{5}{x+1} - \frac{6}{(x+1)^2} + \frac{5}{x^2+1}\right)$$

(13) $p(x) = 27x^6 - 9x^4 + 3x^2 - 1 = -(1 - 3x^2 + 9x^4 - 27x^6)$

$$= -\frac{1-(-3x^2)^4}{1-(-3x^2)} = \frac{81x^8-1}{3x^2+1} = 0 \iff 81x^8 - 1 = 0 \land 3x^2+1 \neq 0$$

$$81x^8 - 1 = 0 \iff x^8 = \frac{1}{81} = \frac{1}{81}e^{0i} \iff x \in \left\{\frac{1}{\sqrt{3}} e^{\frac{k\pi}{4}i} \mid k \in \{-3,-2,\dots,4\}\right\}$$

$$\iff x \in \left\{\frac{1}{\sqrt{3}}e^{-\frac{3\pi}{4}i}, \frac{1}{\sqrt{3}}e^{-\frac{\pi}{2}i}, \frac{1}{\sqrt{3}}e^{-\frac{\pi}{4}i}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}e^{\frac{\pi}{4}i}, \frac{1}{\sqrt{3}}e^{\frac{\pi}{2}i}, \frac{1}{\sqrt{3}}e^{\frac{3\pi}{4}i}, -\frac{1}{\sqrt{3}}\right\}$$

$$3x^2+1 = 0 \iff x^2 = -\frac{1}{3} \iff x \in \left\{-\frac{1}{\sqrt{3}}i, \frac{1}{\sqrt{3}}i\right\} = \left\{\frac{1}{\sqrt{3}}e^{-\frac{\pi}{2}i}, \frac{1}{\sqrt{3}}e^{\frac{\pi}{2}i}\right\}$$

- Faktorizacija nad $\mathbb{C}$:

$$p(x) = 27\left(x - \frac{1}{\sqrt{3}}e^{-\frac{3\pi}{4}i}\right)\left(x - \frac{1}{\sqrt{3}}e^{-\frac{\pi}{4}i}\right)\left(x - \frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{\sqrt{3}}e^{\frac{\pi}{4}i}\right)\left(x - \frac{1}{\sqrt{3}}e^{\frac{3\pi}{4}i}\right)\left(x + \frac{1}{\sqrt{3}}\right)$$

- Faktorizacija nad $\mathbb{R}$:

$$\begin{aligned} (x - re^{\varphi i})\cdot(x - re^{-\varphi i}) &= x^2 - rx(e^{\varphi i} + e^{-\varphi i}) + r^2 e^{\varphi i}\cdot e^{-\varphi i} \\ &= x^2 - 2r\cos\varphi\, x + r^2 \end{aligned}$$

$$\left(x - \frac{1}{\sqrt{3}}e^{-\frac{3\pi}{4}i}\right)\cdot\left(x - \frac{1}{\sqrt{3}}e^{\frac{3\pi}{4}i}\right) = x^2 - 2\cdot\frac{1}{\sqrt{3}}\cos\frac{3\pi}{4}x + \frac{1}{3} = x^2 + \sqrt{\frac{2}{3}}x + \frac{1}{3}$$

$$\left(x - \frac{1}{\sqrt{3}}e^{-\frac{\pi}{4}i}\right)\cdot\left(x - \frac{1}{\sqrt{3}}e^{\frac{\pi}{4}i}\right) = x^2 - 2\cdot\frac{1}{\sqrt{3}}\cos\frac{\pi}{4}x + \frac{1}{3} = x^2 - \sqrt{\frac{2}{3}}x + \frac{1}{3}$$

$$p(x) = 27\left(x^2 + \sqrt{\frac{2}{3}}x + \frac{1}{3}\right)\left(x^2 - \sqrt{\frac{2}{3}}x + \frac{1}{3}\right)\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right)$$

- Faktorizacija nad $\mathbb{Q}$:

$$\left(x^2 + \sqrt{\frac{2}{3}}x + \frac{1}{3}\right)\left(x^2 - \sqrt{\frac{2}{3}}x + \frac{1}{3}\right) = \left(x^2 + \frac{1}{3}\right)^2 - \left(\sqrt{\frac{2}{3}}x\right)^2 = x^4 + \cancel{\frac{2}{3}x^2} + \frac{1}{9} - \cancel{\frac{2}{3}x^2} = x^4 + \frac{1}{9}$$

$$\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right) = x^2 - \frac{1}{3}$$

$$p(x) = 27\left(x^4 + \frac{1}{9}\right)\left(x^2 - \frac{1}{3}\right) = (9x^4+1)(3x^2-1)$$

(14) $p(x) = x^8 + x^4 + 1$

$q(x) = x^2 + x + 1$

I način:

$q(x) \mid p(x) \Rightarrow$ koreni polinoma q su koreni i polinoma p

$x^2+x+1=0 \Rightarrow x_{1,2} = \frac{-1\pm\sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i = e^{\pm\frac{2\pi}{3}i}$

$$p(e^{\pm\frac{2\pi}{3}i}) = (e^{\pm\frac{2\pi}{3}i})^8 + (e^{\pm\frac{2\pi}{3}i})^4 + 1 = e^{\frac{16\pi}{3}i} + e^{\frac{8\pi}{3}i} + 1$$

$$= e^{6\pi i}\cdot e^{-\frac{2\pi}{3}i} + e^{2\pi i}\cdot e^{\frac{2\pi}{3}i} + 1 = e^{-\frac{2\pi}{3}i} + e^{\frac{2\pi}{3}i} + 1$$

$$= 2\cos\frac{2\pi}{3} + 1 = -1+1 = 0$$

II način:

$$(x^8 + x^4 + 1) : (x^2+x+1) = x^6 - x^5 + x^3 - x + 1$$

$$\begin{array}{l}
-x^8 - x^7 - x^6 \\
\hline
\quad -x^7 - x^6 \\
\quad x^7 + x^6 + x^5 \\
\hline
\qquad x^5 + x^4 \\
\qquad -x^5 - x^4 - x^3 \\
\hline
\qquad\quad -x^3 \\
\qquad\quad +x^3 + x^2 + x \\
\hline
\qquad\qquad x^2 + x + 1 \\
\qquad\qquad -x^2 - x - 1 \\
\hline
\qquad\qquad\qquad 0
\end{array}$$

$p(x) = q(x)\cdot(x^6 - x^5 + x^3 - x + 1)$

III način:

$$x^8 + x^4 + 1 = x^8 + 2x^4 + 1 - x^4 = (x^4+1)^2 - (x^2)^2 = (x^4+1-x^2)(x^4+1+x^2)$$

$$= (x^4 - x^2 + 1)(x^4 + 2x^2 + 1 - x^2) = (x^4 - x^2 + 1)((x^2+1)^2 - x^2)$$

$$= (x^4 - x^2 + 1)(x^2 - x + 1)\boxed{(x^2+x+1)}$$

IV način:

$$x^8 + x^4 + 1 = \frac{(x^4)^3 - 1}{x^4 - 1} = \frac{x^{12}-1}{x^4-1} = \frac{(x^6-1)(x^6+1)}{(x^2-1)(x^2+1)} = \frac{((x^3)^2-1)((x^2)^3+1)}{(x-1)(x+1)(x^2+1)}$$

$$= \frac{(x^3-1)(x^3+1)\cancel{(x^2+1)}(x^4-x^2+1)}{(x-1)(x+1)\cancel{(x^2+1)}} = \frac{\cancel{(x-1)}(x^2+x+1)\cancel{(x+1)}(x^2-x+1)(x^4-x^2+1)}{\cancel{(x-1)}\cancel{(x+1)}}$$

$$= \boxed{(x^2+x+1)}(x^2-x+1)(x^4-x^2+1)$$