

SISTEMI LINEARNIH JEDNAČINA

1. Gausovim algoritmom rešiti sistem:

$$\begin{array}{lcl} \begin{array}{rcl} x & + & y & + & 2z & = & 2 \\ \text{(a)} & 2x & - & 3y & - & z & = & 5 \\ & 3x & - & 2y & + & z & = & 10 \end{array} & ; & \begin{array}{rcl} x & - & y & - & z & - & u & = & 3 \\ \text{(b)} & 2x & + & y & - & 3z & + & 2u & = & 1 \\ & -x & + & 2y & & & - & u & = & 0 \end{array} \end{array}$$

$$\begin{array}{lcl} \begin{array}{rcl} x & + & 2y & + & z & = & 4 \\ \text{(c)} & 2x & - & y & - & z & = & 0 \\ & -2x & + & y & - & 3z & = & -4 \\ & 5x & + & 5y & + & 6z & = & 16 \end{array} & ; & \begin{array}{rcl} x & - & y & - & 2z & = & -3 \\ \text{(d)} & -2x & + & 2y & + & 4z & = & 6 \\ & 3x & - & 3y & - & 6z & = & -9 \end{array} \end{array}$$

2. U zavisnosti od realnog parametra a diskutovati i rešiti sistem jednačina:

$$\begin{array}{lcl} \begin{array}{rcl} x & + & y & + & z & = & a \\ \text{(a)} & x & + & (a+1)y & + & z & = & 2a \\ & x & + & y & + & az & = & -a \end{array} & ; & \begin{array}{rcl} x & + & y & + & az & = & 1 \\ \text{(b)} & x & + & ay & + & z & = & a \\ & ax & + & y & + & z & = & a^2 \\ & ax & + & ay & + & az & = & a^3 \end{array} \end{array}$$

3. U zavisnosti od realnih parametara a i b diskutovati i rešiti sistem jednačina:

$$\begin{array}{rcl} x & - & (a+1)y & + & (a+1)z & = & 1 \\ ax & & + & ay & + & (a+1)z & = & 2 \\ ax & & - & 2y & + & (a+1)z & = & b \end{array}$$

4. U zavisnosti od realnih parametara p i q diskutovati i rešiti sistem jednačina:

$$\begin{array}{rcl} px & + & (2p-p^2)y & = & q \\ (p+1)x & + & (p^2-2p)y & = & 3 \\ (2p+1)x & + & (p^2-2p)y & = & 5 \end{array}$$

5. U zavisnosti od realnih parametara a, b i c diskutovati sistem jednačina:

$$\begin{array}{rcl} a(a+1)x & + & y & - & z & + & au & = & 1 \\ a(a+1)x & + & ay & & & + & 2au & = & b \\ (a-1)y & + & (a-1)^2z & + & 3au & = & c-1 \end{array}$$

6. U zavisnosti od parametara $a, b \in \mathbb{C}$ diskutovati sistem jednačina:

$$\begin{array}{rcl} x & + & iy & + & (1+i)z & = & b \\ 2ix & + & ay & + & (2i-4)z & = & (2+2i)b + 2i \\ x & + & (a+2+i)y & + & (i-1)z & = & 4b-1 \end{array}$$

Odrediti skup rešenja sistema u slučaju kada je $a = -2$.

Rešenja:

① (a)
$$\begin{array}{r} x + y + 2z = 2 \quad / \cdot (-2) \quad / \cdot (-3) \\ 2x - 3y - z = 5 \quad \swarrow + \\ 3x - 2y + z = 10 \quad \swarrow + \\ \hline x + y + 2z = 2 \end{array}$$

$$\begin{array}{r} -5y - 5z = 1 \quad / \cdot (-1) \\ -5y - 5z = 4 \quad \swarrow + \\ \hline \end{array}$$

$$\begin{array}{r} x + y + 2z = 2 \\ -5y - 5z = 1 \\ 0 = 3 \end{array}$$

sistem je nemoguć

(b)
$$\begin{array}{r} x - y - z - u = 3 \quad / \cdot (-3) \\ 2x + y - 3z + 2u = 1 \quad \swarrow + \\ -x + 2y - u = 0 \end{array}$$

$$\begin{array}{r} x - y - z - u = 3 \\ -x + 4y + 5u = -8 \quad \swarrow + \\ -x + 2y - u = 0 \quad / \cdot (-1) \end{array}$$

$$\begin{array}{r} z + x - y - u = 3 \Rightarrow z = -5u - 7 \\ -x + 2y - u = 0 \Rightarrow x = -7u - 8 \\ 2y + 6u = -8 \Rightarrow y = -3u - 4 \end{array}$$

$$R = \{(-7u - 8, -3u - 4, -5u - 7, u) \mid u \in \mathbb{R}\}$$

sistem je jednostruko neodređen

(c)
$$\begin{array}{r} x + 2y + z = 4 \quad / \cdot (-2) \quad / \cdot 2 \quad / \cdot (-5) \\ 2x - y - z = 0 \quad \swarrow + \\ -2x + y - 3z = -4 \quad \swarrow + \\ \hline 5x + 5y + 6z = 16 \end{array}$$

$$\begin{array}{r} x + 2y + z = 4 \\ -5y - 3z = -8 \quad \swarrow + \\ 5y - z = 4 \quad \swarrow + \\ -5y + z = -4 \quad \swarrow + \\ \hline \end{array}$$

$$\begin{array}{r} x + 2y + z = 4 \Rightarrow x = 1 \\ 5y - z = 4 \Rightarrow y = 1 \\ -4z = -4 \Rightarrow z = 1 \end{array}$$

$$0 = 0$$

$$(x, y, z) = (1, 1, 1) \Rightarrow \text{sistem je određen}$$

(d)
$$\begin{array}{r} x - y - 2z = -3 \quad / \cdot 2 \quad / \cdot (-3) \\ -2x + 2y + 4z = 6 \quad \swarrow + \\ 3x - 3y - 6z = -9 \quad \swarrow + \\ \hline \end{array}$$

$$\begin{array}{r} x - y - 2z = -3 \\ 0 = 0 \\ 0 = 0 \end{array}$$

$$R = \{(y + 2z - 3, y, z) \mid y, z \in \mathbb{R}\}$$

Sistem je dvostruko neodređen

② (a)
$$\begin{array}{r} x + y + z = a \quad / \cdot (-1) \\ x + (a+1)y + z = 2a \quad \swarrow + \\ x + y + az = -a \quad \swarrow + \\ \hline \end{array}$$

$$\begin{array}{r} x + y + z = a \\ ay = a \\ (a-1)z = -2a \end{array}$$

$$z = \frac{-2a}{a-1}, \quad a \neq 1$$

$$y = \frac{a}{a} = 1, \quad a \neq 0$$

$$x = a - y - z = a - 1 + \frac{2a}{a-1}$$

$$x = \frac{a^2+1}{a-1}, \quad a \neq 1$$

• $a \neq 0 \wedge a \neq 1 \Rightarrow$ sistem je određen

$$(x, y, z) = \left(\frac{a^2+1}{a-1}, 1, -\frac{2a}{a-1} \right)$$

• $\boxed{a=0}$

$$\begin{array}{r} x + y + z = 0 \\ 0 = 0 \\ -z = 0 \end{array}$$

$$z = 0$$

$$y = -x$$

$$R = \{(x, -x, 0) \mid x \in \mathbb{R}\}$$

sistem je jednostruko neodređen

• $\boxed{a=1}$

$$\begin{array}{r} x + y + z = 1 \\ y = 1 \\ 0 = -2 \end{array}$$

Sistem je nemoguć

$$(b) \quad \begin{array}{l} x + y + az = 1 \quad / \cdot (-1) \quad / \cdot (-a) \\ x + ay + z = a \quad \swarrow + \\ ax + y + z = a^2 \quad \swarrow + \\ ax + ay + az = a^3 \quad \swarrow + \end{array}$$

$$x + y + az = 1$$

$$(a-1)y + (1-a)z = a-1$$

$$(1-a)y + (1-a^2)z = a^2 - a$$

$$(a-a^2)z = a^3 - a$$

$$x + y + az = 1$$

$$(a-1)y + (1-a)z = a-1$$

$$(2-a-a^2)z = a^2 - 1$$

$$(a-a^2)z = a^3 - a$$

$$z = \frac{a^2-1}{2-a-a^2} = \frac{(a+1)(a-1)}{-(a-1)(a+2)} = -\frac{a+1}{a+2}, \quad a \neq 1 \text{ i } a \neq -2$$

$$z = \frac{a^3-a}{a-a^2} = \frac{a(a-1)(a+1)}{-a(a-1)} = -(a+1), \quad a \neq 0 \text{ i } a \neq 1$$

$$\frac{a+1}{a+2} = a+1 \Leftrightarrow a = -1$$

• $a \neq 1, a \neq -2, a \neq 0, a \neq -1 \Rightarrow$ sistem je nemoguć

• $|a=1|$

$$x + y + z = 1$$

$$0 = 0$$

$$0 = 0$$

$$0 = 0$$

$$R = \{(1-y-z, y, z) \mid y, z \in \mathbb{R}\}$$

sistem je dvostruko neodređen

• $|a=-2|$

$$x + y - 2z = 1$$

$$y - z = 1$$

$$0 = 3$$

sistem je nemoguć

• $|a=-1|$

$$x + y - z = 1 \Rightarrow x = 0$$

$$y - z = 1 \Rightarrow y = 1$$

$$2z = 0 \Rightarrow z = 0$$

$$-2z = 0 \Rightarrow z = 0$$

$$(x, y, z) = (0, 1, 0)$$

sistem je određen

• $|a=0|$

$$x + y = 1 \Rightarrow x = \frac{1}{2}$$

$$y - z = 1 \Rightarrow y = \frac{1}{2}$$

$$2z = -1 \Rightarrow z = -\frac{1}{2}$$

$$0 = 0$$

$$(x, y, z) = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$$

sistem je određen

$$(3) \quad \begin{array}{l} x - (a+1)y + (a+1)z = 1 \quad / \cdot (-1) \\ ax + ay + (a+1)z = 2 \quad \swarrow + \\ ax - 2y + (a+1)z = b \quad \swarrow + \end{array}$$

$$ax + ay + (a+1)z = 2$$

$$ax - 2y + (a+1)z = b$$

$$(a+1)z + x - (a+1)y = 1$$

$$(a-1)x + (2a+1)y = 1 \quad / \cdot (-1)$$

$$(a-1)x + (a-1)y = b-1 \quad \swarrow +$$

$$(a+1)z + x - (a+1)y = 1$$

$$(a-1)x + (2a+1)y = 1$$

$$-(a+2)y = b-2$$

$$y = \frac{2-b}{a+2}, \quad a \neq -2$$

$$(a-1)x = 1 - (2a+1)y$$

$$\Rightarrow x = \frac{-3a+2ab+b}{(a+2)(a-1)}, \quad a \neq -2, a \neq 1$$

$$(a+1)z = 1 - x + (a+1)y$$

$$\Rightarrow z = \frac{3a^2+4a-2ab-a^2b-4}{(a+2)(a-1)(a+1)}$$

$$a \neq -2, a \neq 1, a \neq -1$$

• $a \in \mathbb{R} \setminus \{-2, 1, -1\} \Rightarrow$ sistem je određen

• $|a=-2|$

$$-z + x + y = 1$$

$$-3x - 3y = 1$$

$$0 = b-2$$

$b \neq 2 \Rightarrow$ sistem je nemoguć

$b = 2 \Rightarrow$ sistem je jednostruko neodređen

$$R = \{(x, -x - \frac{1}{3}, -\frac{4}{3}) \mid x \in \mathbb{R}\}$$

• $|a=1|$

$$2z + x - 2y = 1$$

$$3y = 1$$

$$-3y = b-2$$

$$2z + x - 2y = 1$$

$$y = \frac{1}{3}$$

$$0 = b-1$$

$b \neq 1 \Rightarrow$ sistem je nemoguć

$b = 1 \Rightarrow$ sistem je jednostruko neodređen

$$R = \{(\frac{5}{3}-2z, \frac{1}{3}, z) \mid z \in \mathbb{R}\}$$

• $|a=-1|$

$$x = 1$$

$$-2x - y = 1$$

$$y = b-2$$

$$x = 1$$

$$y = -3$$

$$-y = b-2 \quad \swarrow +$$

$$x = 1$$

$$y = -3$$

$$0 = b-5$$

$b \neq 5 \Rightarrow$ sistem je nemoguć

$b = 5 \Rightarrow$ sistem je jednostruko neodređen

$$R = \{(1, -3, z) \mid z \in \mathbb{R}\}$$

$$\begin{aligned}
 (4) \quad & px + (2p - p^2)y = 2 \\
 & (p+1)x + (p^2 - 2p)y = 3 \\
 & (2p+1)x + (p^2 - 2p)y = 5 \\
 & \begin{aligned} & (p^2 - 2p)y + (p+1)x = 3 \quad / \cdot (-1) \\ & + \quad (p^2 - 2p)y + (2p+1)x = 5 \quad \checkmark \\ & - \quad (p^2 - 2p)y \quad + px = 2 \end{aligned}
 \end{aligned}$$

$$\begin{aligned}
 & (p^2 - 2p)y + (p+1)x = 3 \\
 & px = 2 \Rightarrow x = \frac{2}{p} \\
 & (2p+1)x = 2 + 3 \Rightarrow x = \frac{2+3}{2p+1}
 \end{aligned}$$

• $p=0 \Rightarrow$ sistem je nemoguć

$p \neq 0$

$$\begin{aligned}
 \frac{2}{p} &= \frac{2+3}{2p+1} \Leftrightarrow p(2+3) = 4p+2 \\
 &\Leftrightarrow pg = p+2 \\
 &\Leftrightarrow g = \frac{p+2}{p}
 \end{aligned}$$

• $p \neq 0$ i $g \neq \frac{p+2}{p} \Rightarrow$ sistem je nemoguć

• $p \neq 0$ i $g = \frac{p+2}{p}$

$$\begin{aligned}
 x = \frac{2}{p} \Rightarrow p(p-2)y &= 3 - (p+1) \cdot \frac{2}{p} \\
 &= \frac{3p - 2p - 2}{p} = \frac{p-2}{p} \\
 y &= \frac{\frac{p-2}{p}}{p^2(p-2)} = \frac{1}{p^2}, \quad p \neq 2
 \end{aligned}$$

$\Rightarrow$ za $p \neq 0, p \neq 2$ i $g = \frac{p+2}{p}$ sistem je određen: $(x, y) = (\frac{2}{p}, \frac{1}{p^2})$

• $p=2$ i $g = \frac{p+2}{p} = 2$

$$\begin{aligned}
 3x &= 3 \\
 2x &= 2 \\
 5x &= 5
 \end{aligned} \Rightarrow x = 1$$

$R = \{(1, y) \mid y \in \mathbb{R}\} \Rightarrow$ sistem je jednostruko neodređen

$$\begin{aligned}
 (5) \quad & a(a+1)x + y - z + au = 1 \quad / \cdot (-1) \\
 & a(a+1)x + ay + 2au = b \quad \checkmark \\
 & (a-1)y + (a-1)z + 3au = c-1
 \end{aligned}$$

$$\begin{aligned}
 & a(a+1)x + y - z + au = 1 \\
 & (a-1)y + z + au = b-1 \quad / \cdot (-1) \\
 & (a-1)y + (a^2 - 2a + 1)z + 3au = c-1 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 & \boxed{a(a+1)}x + y - z + au = 1 \\
 & \boxed{(a-1)}y + z + au = b-1 \\
 & \boxed{a(a-2)}z + 2au = c-b
 \end{aligned}$$

• $\boxed{a \in \mathbb{R} \setminus \{0, 2, 1, -1\}}$

$$R = \{(x(u), y(u), z(u), u) \mid u \in \mathbb{R}\}$$

sistem je jednostruko neodređen

• $\boxed{a=0}$

$$\begin{aligned}
 y - z &= 1 \\
 -y + z &= b-1 \\
 0 &= c-b
 \end{aligned}$$

$$y = z + 1$$

$$0 = b$$

$$c = b$$

$$- \underline{c=b=0}$$

$$R = \{(x, z+1, z, u), x, z, u \in \mathbb{R}\}$$

Sistem je trostruko neodređen

$$- \underline{b \neq 0 \vee c \neq b}$$

sistem je nemoguć

• $\boxed{a=2}$

$$\begin{aligned}
 6x + y - z + 2u &= 1 \\
 y + z + 2u &= b-1
 \end{aligned}$$

$$4u = c-b$$

$$R = \{(x(z), y(z), z, \frac{c-b}{4}) \mid z \in \mathbb{R}\}$$

sistem je jednostruko neodređen

• $\boxed{a=1}$

$$\begin{aligned}
 2x + y - z + u &= 1 \\
 z + u &= b-1 \\
 -z + 2u &= c-b
 \end{aligned}$$

$$\begin{aligned}
 y &= 1 - 2x + z - u \\
 z &= b-1-u \Rightarrow z = \frac{3b-c-2}{3} \\
 u &= \frac{c-1}{3}
 \end{aligned}$$

$$R = \{(x, y(x), \frac{3b-c-2}{3}, \frac{c-1}{3}) \mid x \in \mathbb{R}\}$$

sistem je jednostruko neodređen

• $\boxed{a=-1}$

$$\begin{aligned}
 y - z - u &= 1 \quad / \cdot (-2) \\
 -2y + z - u &= b-1 \quad \checkmark + \\
 3z - 2u &= c-b
 \end{aligned}$$

$$y - z - u = 1$$

$$3z + u = b-3 \quad / \cdot (-1)$$

$$3z - 2u = c-b \quad \checkmark +$$

$$\begin{aligned}
 y - z - u &= 1 \\
 3z + u &= b-3
 \end{aligned}$$

$$-3u = c-2b+3$$

sistem je jednostruko neodređen

$x \in \mathbb{R}, y, z$ i u su jednoznačno određeni

$$\begin{array}{rcl} \textcircled{6} & x & + iy + (1+i)z = b \quad / \cdot (-2i) \\ 2ix & + ay + (2i-4)z = (2+2i)b + 2i & \swarrow + \\ & x + (a+2+i)y + (i-1)z = 4b-1 & \leftarrow + \end{array}$$

$$\begin{array}{rcl} x & + iy + (1+i)z = b \\ (a+2)y & - 2z = 2b + 2i & / \cdot (-1) \\ (a+2)y & - 2z = 3b-1 & \swarrow + \end{array}$$

$$\begin{array}{rcl} x & + iy + (1+i)z = b \\ (a+2)y & - 2z = 2b + 2i \\ & 0 = b-1-2i \end{array}$$

• $b \neq 1+2i \Rightarrow$ sistem je nemoguć!

• $b = 1+2i$

$$\begin{array}{rcl} x & + iy + (1+i)z = 1+2i \\ (a+2)y & - 2z = 2+6i \end{array}$$

$$x = 1+2i - iy - (1+i)z$$

$$y = \frac{2z + 2 + 6i}{a+2}, \quad a \neq -2$$

- $a \neq -2$

$$R = \{(x(z), y(z), z) \mid z \in \mathbb{R}\}$$

sistem je jednostruko neodređen

- $a = -2$

$$\begin{array}{rcl} x & + iy + (1+i)z = 1+2i \\ & - 2z = 2+6i \end{array}$$

$$z = -1-3i$$

$$\begin{aligned} x &= 1+2i - iy + (1+i)(-1-3i) \\ &= 1+2i - iy + 1+4i-3 \\ &= 6i-1-iy = -1+(6-y)i \end{aligned}$$

sistem je jednostruko neodređen

$$\bullet \boxed{a = -2}$$

$$- b \neq 1+2i \Rightarrow R = \emptyset$$

$$- b = 1+2i$$

$$R = \{(-1+(6-y)i, y, -1-3i) \mid y \in \mathbb{R}\}$$