# VEKTORSKI PROSTORI

1. (1) Dokazati da je $W = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 = 0\}$ potprostor vektorskog prostora $(\mathbb{R}^3, \mathbb{R}, +, \cdot)$.
   (2) Dokazati da $W = \{(x, y, z) \mid x, y, z \in \mathbb{Q}\}$ nije potprostor vektorskog prostora $(\mathbb{R}^3, \mathbb{R}, +, \cdot)$.
   (3) Dokazati da je $W = \{(x, y, x + y) \mid x, y \in \mathbb{R}\}$ potprostor vektorskog prostora $(\mathbb{R}^3, \mathbb{R}, +, \cdot)$.
   (4) Da li regularne matrice formata $2 \times 2$ nad poljem realnih brojeva, čine potprostor vektorskog prostora svih matrica formata $2 \times 2$, nad poljem realnih brojeva?

2. Dati su vektori

   (a) $a_1 = (4, 4, 3)$, $a_2 = (7, 2, 1)$, $a_3 = (4, 1, 6)$ i $b = (5, 9, \lambda)$
   (b) $a_1 = (2, 1, 0)$, $a_2 = (-3, 2, 1)$, $a_3 = (5, -1, -1)$ i $b = (8, \lambda, -2)$
   (c) $a_1 = (-1, 3, -4)$, $a_2 = (1, -3, 4)$, $a_3 = (2, -6, 8)$ i $b = (0, \lambda, -1)$.

   Odrediti $\lambda \in \mathbb{R}$ tako da se vektor $b$ može izraziti kao linearna kombinacija vektora $a_1, a_2$ i $a_3$.

3. Ispitati linearnu zavisnost skupova vektora:
   $A = \{(-4, 2, -1, 3), (1, -3, 2, 4), (-2, 4, 3, -1), (-3, 5, 1, -2)\}$
   $B = \{(1, 1, 2, 1), (1, -1, 1, 2), (-3, 1, -4, -5), (0, 2, 1, -1)\}$
   $C = \{1 + x,\ x,\ 2x^2,\ 1 - x + x^2\}$
   $D = \{3, \sin^2 x, \cos^2 x\}$

4. Dati su vektori $a_1 = (3, 1, 1)$, $a_2 = (m, -1, 0)$ i $a_3 = (0, 1, m)$

   (a) Za koje vrednosti realnog parametra $m$ skup $\{a_1, a_2, a_3\}$ predstavlja bazu prostora $\mathbb{R}^3$?
   (b) Ako je $m = 2$ napisati vektor $b = (4, 6, 8)$ kao linearnu kombinaciju vektora $a_1, a_2, a_3$.

5. Skup vektora $A = \{x, y, u, v\}$ čini bazu vektorskog prostora $\mathbb{R}^4$. Da li je i skup vektora $B = \{x + u,\ 2y + v,\ x + u - v,\ y - 3u\}$ baza tog prostora?

6. Neka je $S$ vektorski prostor generisan skupom $A = \{a, b, c, d, e\}$ gde su

   $$a = (3, 3, 0, 6, 9),\ b = (0, 2, 1, 0, 4),\ c = (1, 1, 2, 1, 4),\ d = (2, 2, 0, 4, 6),\ e = (2, 0, 1, 3, 3).$$

   (a) Odrediti dimenziju vektorskog prostora $S$ i linearnu zavisnost medju vektorima skupa $A$.
   (b) Odrediti sve podskupove skupa $A$ koji su baze vektorskog prostora $S$.
   (c) Proveriti da li vektori $x = (1, 1, 0, 2, 1)$ i $y = (0, 4, 0, 1, 7)$ pripadaju prostoru $S$.

7. Odrediti dimenziju vektorskog prostora $V$ generisanog skupom vektora $A = \{a, b, c, d, e\}$ i naći sve podskupove skupa $A$ koji su baze prostora $V$ ako su sve zavisnosti medju vektorima skupa $A$ date jednačinama

   $$\begin{array}{rcrcrcrcrcl} a & + & 2b & + & 4c & - & d & + & e & = & 0 \\ 2a & + & 3b & + & c & + & 5d & + & 2e & = & 0 \\ a & + & 2b & & & + & 3d & + & e & = & 0 \\ a & + & 2b & + & 2c & + & d & + & e & = & 0 \end{array}$$

8. Vektorski prostor $V$ generisan je vektorima

   $$v_1 = (a, 1, 1),\ \ v_2 = (-a, a, -a^2),\ \ v_3 = (a^3, -a, 1).$$

   Naći njegovu dimenziju i bazu u zavisnosti od realnog parametra $a$.

9. Pokazati da je $S = \{(a, b, c, d) \in \mathbb{R}^4 \mid a + b = c + d\}$ vektorski potprostor prostora $\mathbb{R}^4$. Odrediti jednu njegovu bazu, a zatim tu bazu dopuniti do baze vektorskog prostora $\mathbb{R}^4$.

10. (a) Pokazati da je $\mathcal{P} = \{p \in \mathbb{R}[x] \mid p(1) = 0,\ st(p) < 4\}$ potprostror vektorskog prostora $\mathbb{R}[x]$.
    (b) Naći $k, l, m \in \mathbb{R}$ tako da $\mathcal{B} = \{\, x + m,\ x^2 + m,\ x^3 + kx^2 + lx + m\}$ bude baza za $\mathcal{P}$.
    (c) Ako je $m = -1$ i $k = 1$ izraziti polinom $p(x) = x^3 + 9x^2 - 7x - 3$ kao linearnu kombinaciju vektora baze.

Rešenja:

① (1) $W = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+z^2=0\} = \{(0,y,0) \mid y \in \mathbb{R}\}$

$v_1 = (0,y_1,0)$

$v_2 = (0,y_2,0)$

$\alpha,\beta \in \mathbb{R}$

$$\begin{aligned}\alpha v_1+\beta v_2 &= \alpha(0,y_1,0)+\beta(0,y_2,0)\\ &= (0,\alpha y_1,0)+(0,\beta y_2,0)\\ &= (0,\alpha y_1+\beta y_2,0) \in W\end{aligned}$$

$\Rightarrow$ W jeste potprostor od $(\mathbb{R}^3,\mathbb{R},+,\cdot)$

(2) $W = \{(x,y,z) \mid x,y,z \in \mathbb{Q}\} = \mathbb{Q}^3$

$v = (1,1,1) \in W$

$\alpha = \sqrt{2} \in \mathbb{R}\setminus\mathbb{Q}$

$\alpha v = \sqrt{2}(1,1,1) = (\sqrt{2},\sqrt{2},\sqrt{2}) \notin W$

$\Rightarrow$ W nije potprostor od $(\mathbb{R}^3,\mathbb{R},+,\cdot)$

(3) $W = \{(x,y,x+y) \mid x,y \in \mathbb{R}\}$

$v_1 = (x_1,y_1,x_1+y_1)$

$v_2 = (x_2,y_2,x_2+y_2)$

$\alpha,\beta \in \mathbb{R}$

$$\begin{aligned}\alpha v_1+\beta v_2 &= \alpha(x_1,y_1,x_1+y_1)+\beta(x_2,y_2,x_2+y_2)\\ &= (\alpha x_1,\alpha y_1,\alpha x_1+\alpha y_1)+(\beta x_2,\beta y_2,\beta x_2+\beta y_2)\\ &= (\alpha x_1+\beta x_2,\ \alpha y_1+\beta y_2,\ (\alpha x_1+\beta x_2)+(\alpha y_1+\beta y_2)) \in W\end{aligned}$$

$\Rightarrow$ W jeste potprostor od $(\mathbb{R}^3,\mathbb{R},+,\cdot)$

(4) $$\left.\begin{aligned} A &= \begin{bmatrix}1&0\\0&-1\end{bmatrix} \to |A| = -1 \neq 0\\ B &= \begin{bmatrix}-1&0\\0&2\end{bmatrix} \to |B| = -2 \neq 0\end{aligned}\right\} \Rightarrow \text{A i B su regularne matrice}$$

$$A+B = \begin{bmatrix}1&0\\0&-1\end{bmatrix}+\begin{bmatrix}-1&0\\0&2\end{bmatrix} = \begin{bmatrix}0&0\\0&1\end{bmatrix} \to |A+B| = 0 \Rightarrow \text{A+B nije regularna matrica}$$

② (a) $a_1 = (4,4,3)$

$a_2 = (7,2,1)$

$a_3 = (4,1,6)$

$b = (5,9,\lambda)$

$$\begin{aligned}&\alpha a_1+\beta a_2+\gamma a_3 = b\\ &\alpha(4,4,3)+\beta(7,2,1)+\gamma(4,1,6) = (5,9,\lambda)\\ &(4\alpha,4\alpha,3\alpha)+(7\beta,2\beta,\beta)+(4\gamma,\gamma,6\gamma) = (5,9,\lambda)\\ &(4\alpha+7\beta+4\gamma,\ 4\alpha+2\beta+\gamma,\ 3\alpha+\beta+6\gamma) = (5,9,\lambda)\end{aligned}$$

$$\begin{aligned}4\alpha+7\beta+4\gamma &= 5\\ 4\alpha+2\beta+\gamma &= 9\\ 3\alpha+\beta+6\gamma &= \lambda\end{aligned}$$

Prema Kroneker-Kapelijevoj t-mi ovaj sistem ima rešenje ako je rang $(M_s)$ = rang $(M_p)$.

$$\left[\begin{array}{ccc|c}4&7&4&5\\4&2&1&9\\3&1&6&\lambda\end{array}\right] \sim \left[\begin{array}{ccc|c}4&2&1&9\\4&7&4&5\\3&1&6&\lambda\end{array}\right] \sim \left[\begin{array}{ccc|c}1&2&4&9\\4&7&4&5\\6&1&3&\lambda\end{array}\right]\cdot(-4)\,/\cdot(-6) \sim \left[\begin{array}{ccc|c}1&2&4&9\\0&-1&-12&-31\\0&-11&-21&\lambda-54\end{array}\right]\cdot(-11)$$

$$\sim \left[\begin{array}{ccc|c}1&2&4&9\\0&-1&-12&-31\\0&0&111&\lambda+287\end{array}\right]$$

rang $(M_s)$ = rang $(M_p)$ = 3

$\Downarrow$

Sistem je određen za sve $\lambda \in \mathbb{R}$

(b) $a_1=(2,1,0)$
$a_2=(-3,2,1)$
$a_3=(5,-1,-1)$
$b=(8,\lambda,-2)$

$$\left[\begin{array}{ccc|c} 2 & -3 & 5 & 8 \\ 1 & 2 & -1 & \lambda \\ 0 & 1 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & \lambda \\ 2 & -3 & 5 & 8 \\ 0 & 1 & -1 & -\lambda \end{array}\right] \cdot(-2) \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & \lambda \\ 0 & -7 & 7 & 8-2\lambda \\ 0 & 1 & -1 & -\lambda \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & \lambda \\ 0 & 1 & -1 & -\lambda \\ 0 & -7 & 7 & 8-2\lambda \end{array}\right] \cdot 7 \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & \lambda \\ 0 & 1 & -1 & -\lambda \\ 0 & 0 & 0 & -6-2\lambda \end{array}\right]$$

$$\text{rang}(M_s)=\text{rang}(M_p)=2 \iff -6-2\lambda=0$$
$$\iff \boxed{\lambda=-3}$$

(c) $a_1=(-1,3,-4)$
$a_2=(1,-3,4)$
$a_3=(2,-6,8)$
$b=(0,\lambda,-1)$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 3 & -3 & -6 & \lambda \\ -4 & 4 & 8 & -1 \end{array}\right] \cdot 3 \;|\cdot(-4) \sim \left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & -1 \end{array}\right]$$

$\text{rang}(M_s)=1$ i $\text{rang}(M_p)=2$, za sve $\lambda\in\mathbb{R}$

$\Rightarrow$ sistem nema rešenje, tj. ne može se $b$ napisati kao linearna kombinacija $a_1$, $a_2$ i $a_3$.

3. • $A=\{(-4,2,-1,3),(1,-3,2,4),(-2,4,3,-1),(-3,5,1,-2)\}$

$$A=\begin{bmatrix} -4 & 1 & -2 & -3 \\ 2 & -3 & 4 & 5 \\ -1 & 2 & 3 & 1 \\ 3 & 4 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 1 \\ 2 & -3 & 4 & 5 \\ -4 & 1 & -2 & -3 \\ 3 & 4 & -1 & -2 \end{bmatrix} \cdot 2 \;|\cdot(-4)\;|\cdot 3 \sim \begin{bmatrix} -1 & 2 & 3 & 1 \\ 0 & 1 & 10 & 7 \\ 0 & -7 & -14 & -7 \\ 0 & 10 & 8 & 1 \end{bmatrix} \cdot \tfrac{1}{7}$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & 1 \\ 0 & 1 & 10 & 7 \\ 0 & -1 & -2 & -1 \\ 0 & 10 & 8 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 1 & 10 & 7 \\ 0 & 10 & 8 & 1 \end{bmatrix} \cdot 10 \sim \begin{bmatrix} -1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 8 & 6 \\ 0 & 0 & -12 & -9 \end{bmatrix} \begin{matrix} \\ \\ \cdot\frac{1}{2} \\ \cdot\frac{1}{3} \end{matrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & -4 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rang}(A)=3<4 \Rightarrow$ vektori su linearno nezavisni

• $B=\{(1,1,2,1),(1,-1,1,2),(-3,1,-4,-5),(0,2,1,-1)\}$

$$B=\begin{bmatrix} 1 & 1 & -3 & 0 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & -4 & 1 \\ 1 & 2 & -5 & -1 \end{bmatrix} \cdot(-1)\;|\cdot(-2) \sim \begin{bmatrix} 1 & 1 & -3 & 0 \\ 0 & -2 & 4 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & -1 & 2 & 1 \\ 0 & -2 & 4 & 2 \end{bmatrix} \cdot 2$$

$$\sim \begin{bmatrix} 1 & 1 & -3 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rang}(B)=2<4 \Rightarrow$ vektori su linearno nezavisni

- $C = \{1+x, x, 2x^2, 1-x+x^2\}$

I način

$$\alpha(1+x)+\beta x+\gamma\cdot 2x^2+\delta(1-x+x^2)=0$$
$$\alpha+\alpha x+\beta x+2\gamma x^2+\delta-\delta x+\delta x^2=0$$
$$(\alpha+\delta)+(\alpha+\beta-\delta)x+(2\gamma+\delta)x^2=0$$

$$\begin{aligned} \alpha \quad\quad &+\delta=0 \;/\cdot(-1) \\ \alpha+\beta \quad &-\delta=0 \\ 2\gamma &-\delta=0 \end{aligned} \iff \begin{aligned} \alpha \quad &+\delta=0 \Rightarrow \alpha=-\delta \\ \beta \quad &-2\delta=0 \Rightarrow \beta=2\delta \\ 2\gamma &+\delta=0 \Rightarrow \gamma=-\tfrac{1}{2}\delta \end{aligned}$$

sistem je neodređen $\Rightarrow$ vektori su lin. zavisni

II način

$1+x=(1,1,0)$

$x=(0,1,0)$

$2x^2=(0,0,2)$

$1-x+x^2=(1,-1,1)$

$$C=\begin{bmatrix}1&0&0&1\\1&1&0&-1\\0&0&2&1\end{bmatrix}$$

$\operatorname{rang}(C)=3 \Rightarrow$ vektori su linearno zavisni

- $D=\{3, \sin^2 x, \cos^2 x\}$

$3=3\sin^2 x+3\cos^2 x \Rightarrow$ vektori su lin. zavisni

4) $a_1=(3,1,1)$

$a_2=(m,-1,0)$

$a_3=(0,1,m)$

(a) Skup $\{a_1,a_2,a_3\}$ će biti baza za $\mathbb{R}^3$ ako je lin. nezavisan.

$\alpha a_1+\beta a_2+\gamma a_3=0$

$\alpha(3,1,1)+\beta(m,-1,0)+\gamma(0,1,m)=(0,0,0)$

$$\begin{aligned} 3\alpha+m\beta \quad &=0 \\ \alpha-\beta+\gamma&=0 \\ \alpha \quad +m\gamma&=0 \end{aligned}$$

$a_1,a_2,a_3$ su lin. nezav. $\iff \alpha=\beta=\gamma=0$

$\iff$ sistem je određen

$\iff$ det. sistema je $\neq 0$

$$D=\begin{vmatrix}3&m&0\\1&-1&1\\1&0&m\end{vmatrix}=\begin{vmatrix}3&m+3&-3\\1&0&0\\1&1&m-1\end{vmatrix}=-\begin{vmatrix}m+3&-3\\1&m-1\end{vmatrix}=-\left((m+3)(m-1)+3\right)$$
$$=-(m^2+2m-\cancel{3}+\cancel{3})=-m(m+2)$$

$\{a_1,a_2,a_3\}$ je baza za $\mathbb{R}^3$ akko $a\in\mathbb{R}\setminus\{0,-2\}$

(b) $m=2$

$a_1=(3,1,1)$

$a_2=(2,-1,0)$

$a_3=(0,1,2)$

$b=(4,6,8)$

$\alpha a_1+\beta a_2+\gamma a_3=b$

$\alpha(3,1,1)+\beta(2,-1,0)+\gamma(0,1,2)=(4,6,8)$

$$\begin{aligned} 3\alpha+2\beta \quad &=4 \\ \alpha-\beta+\gamma&=6 \\ \alpha \quad +2\gamma&=8 \end{aligned} \iff (\alpha,\beta,\gamma)=(2,-1,3) \Rightarrow b=2a_1-a_2+3a_3$$

5. $A = \{x, y, u, v\}$ baza za $\mathbb{R}^4$

$B = \{x+u,\ 2y+v,\ x+u-v,\ y-3u\}$ će biti baza za $\mathbb{R}^4$ ako je lin. nezavisan

$$\alpha(x+u) + \beta(2y+v) + \gamma(x+u-v) + \delta(y-3u) = 0$$

$$\alpha x + \alpha u + 2\beta y + \beta v + \gamma x + \gamma u - \gamma v + \delta y - 3\delta u = 0$$

$$(\alpha+\gamma)x + (2\beta+\delta)y + (\alpha+\gamma-3\delta)u + (\beta-\gamma)v = 0$$

$x, y, u, v$ su lin. nezavisni

$$\begin{array}{l} \alpha + \gamma = 0 \\ 2\beta + \delta = 0 \\ \alpha + \gamma - 3\delta = 0 \\ \beta - \gamma = 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \alpha + \gamma = 0 & \Rightarrow \alpha = 0 \\ 2\beta + \delta = 0 & \Rightarrow \beta = 0 \\ -3\delta = 0 & \Rightarrow \delta = 0 \\ \beta - \gamma = 0 & \Rightarrow \gamma = 0 \end{array}$$

$\Rightarrow$ B je lin. nezavisan skup vektora, pa je baza za $\mathbb{R}^4$.

6. (a)

$a = (3,3,0,6,9)$
$b = (0,2,1,0,4)$
$c = (1,1,2,1,4)$
$d = (2,2,0,4,6)$
$e = (2,0,1,3,3)$

Dimenzija prostora generisanog vektorima $a, b, c, d, e$ jednaka je rangu matrice čije su kolone dati vektori

$$A = \begin{bmatrix} 3 & 0 & 1 & 2 & 2 \\ 3 & 2 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 6 & 0 & 1 & 4 & 3 \\ 9 & 4 & 4 & 6 & 3 \end{bmatrix} \cdot(-1)\ |\cdot(-2)\ |\cdot(-3) \sim \begin{bmatrix} 3 & 0 & 1 & 2 & 2 \\ 0 & 2 & 0 & 0 & -2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 4 & 1 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 4 & 1 & 0 & -3 \end{bmatrix} \cdot(-2)\ |\cdot(-4)$$

$$\sim \begin{bmatrix} 3 & 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & -4 & 0 & -4 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -7 & 0 & -7 \end{bmatrix} \begin{array}{l} \\ \\ \cdot(-\frac{1}{4}) \\ \\ \cdot\frac{1}{7} \end{array} \sim \begin{bmatrix} 3 & 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rang}(A) = 3 \Rightarrow \dim(S) = 3$$

$$\alpha a + \beta b + \gamma c + \delta d + \varepsilon e = 0$$

$$\begin{array}{l} 3\alpha + \gamma + 2\delta + 2\varepsilon = 0 \\ 3\alpha + 2\beta + \gamma + 2\delta = 0 \\ \beta + 2\gamma + \varepsilon = 0 \\ 6\alpha + \gamma + 4\delta + 3\varepsilon = 0 \\ 9\alpha + 4\beta + 4\gamma + 6\delta + 3\varepsilon = 0 \end{array}$$

$$\begin{array}{l} 3\alpha + \gamma + 2\delta + 2\varepsilon = 0 \\ \beta + 2\gamma + \varepsilon = 0 \\ \gamma + \varepsilon = 0 \ |\cdot(-2)\ |\cdot(-1) \end{array}$$

$$\begin{array}{ll} 3\alpha + 2\delta + \varepsilon = 0 & \Rightarrow \alpha = -\frac{2}{3}\delta - \frac{1}{3}\varepsilon \\ \beta - \varepsilon = 0 & \Rightarrow \beta = \varepsilon \\ \gamma + \varepsilon = 0 & \Rightarrow \gamma = -\varepsilon \end{array}$$

$\delta = 0,\ \varepsilon = -3 \rightarrow \alpha = 1,\ \beta = -3,\ \gamma = 3$

$\delta = -3,\ \varepsilon = 0 \rightarrow \alpha = 2,\ \beta = 0,\ \gamma = 0$

$$\boxed{\begin{array}{l} a - 3b + 3c - 3e = 0 \\ 2a - 3d = 0 \end{array}}$$

(b) $\begin{aligned} a-3b+3c\phantom{-3d}-3e&=0\\ 2a\phantom{-3b+3c}-3d\phantom{-3e}&=0\end{aligned}$

Pošto je $\dim(S)=3$, proveravamo koji od troelementnih podskupova skup $A$ mogu biti baze za $S$.

$\{a,b,c\}\ \checkmark \begin{array}{c} \begin{matrix} d & e \end{matrix} \\ \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} \end{array} = -9$ $\quad\{a,c,e\}\ \checkmark \begin{array}{c} \begin{matrix} b & d \end{matrix} \\ \begin{vmatrix} -3 & 0 \\ 0 & -3 \end{vmatrix} \end{array} = 9$ $\quad\{b,d,e\}\ \checkmark \begin{array}{c} \begin{matrix} a & c \end{matrix} \\ \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} \end{array} = -6$

$\{a,b,d\}\ \times \begin{array}{c} \begin{matrix} c & e \end{matrix} \\ \begin{vmatrix} 3 & -3 \\ 0 & 0 \end{vmatrix} \end{array} = 0$ $\quad\{a,d,e\}\ \times \begin{array}{c} \begin{matrix} b & c \end{matrix} \\ \begin{vmatrix} -3 & 3 \\ 0 & 0 \end{vmatrix} \end{array} = 0$ $\quad\{c,d,e\}\ \checkmark \begin{array}{c} \begin{matrix} a & b \end{matrix} \\ \begin{vmatrix} 1 & -3 \\ 2 & 0 \end{vmatrix} \end{array} = 6$

$\{a,b,e\}\ \checkmark \begin{array}{c} \begin{matrix} c & d \end{matrix} \\ \begin{vmatrix} 3 & 0 \\ 0 & -3 \end{vmatrix} \end{array} = -9$ $\quad\{b,c,d\}\ \checkmark \begin{array}{c} \begin{matrix} a & e \end{matrix} \\ \begin{vmatrix} 1 & -3 \\ 2 & 0 \end{vmatrix} \end{array} = 6$

$\{a,c,d\}\ \times \begin{array}{c} \begin{matrix} b & e \end{matrix} \\ \begin{vmatrix} -3 & -3 \\ 0 & 0 \end{vmatrix} \end{array} = 0$ $\quad\{b,c,e\}\ \checkmark \begin{array}{c} \begin{matrix} a & d \end{matrix} \\ \begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} \end{array} = -3$

(c) $x=(1,1,0,2,1)$

$$\left[\begin{array}{ccccc|c} 3&0&1&2&2&1\\ 3&2&1&2&0&1\\ 0&1&2&0&1&0\\ 6&0&1&4&3&2\\ 9&4&4&6&3&1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 3&0&1&2&2&1\\ 0&1&2&0&1&0\\ 0&0&1&0&1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&-2 \end{array}\right]$$

$\left.\begin{array}{l} \mathrm{rang}(M_s)=3 \\ \mathrm{rang}(M_p)=4 \end{array}\right\} \Rightarrow$ vektor $x$ se ne može napisati kao lin. kombinacija vektora iz $A$, tj $x\notin S$

$y=(0,4,0,1,7)$

$$\left[\begin{array}{ccccc|c} 3&0&1&2&2&0\\ 3&2&1&2&0&4\\ 0&1&2&0&1&0\\ 6&0&1&4&3&1\\ 9&4&4&6&3&7 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 3&0&1&2&2&0\\ 0&1&2&0&1&0\\ 0&0&1&0&1&-1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{array}\right]$$

$\left.\begin{array}{l} \mathrm{rang}(M_s)=3 \\ \mathrm{rang}(M_p)=3 \end{array}\right\} \Rightarrow y\in S$

(7)

$$\begin{array}{l} a+2b+4c-d+e=0 \ /\cdot(-2)\ /\cdot(-1)\\ 2a+3b+c+5d+2e=0\\ a+2b+3d+e=0\\ a+2b+2c+d+e=0 \end{array}$$

$$\begin{array}{l} a+2b+4c-d+e=0\\ -b-7c+7d=0\\ -4c+4d=0\ /\cdot(-\tfrac{1}{4})\\ -2c+2d=0\ /\cdot\tfrac{1}{2} \end{array}$$

$$\begin{array}{l} a+2b+4c-d+e=0\\ -b-7c+7d=0\\ c-d=0 \quad \cdot 7\\ -c+d=0 \end{array}$$

$$\begin{array}{l} a+2b+3c+e=0\\ -b=0\\ c-d=0 \end{array}$$

$b=0$

$\boxed{c=d}$

$a=-2b-3c-e \Rightarrow \boxed{a=-3d-e}$

$\{d,e\}$ je jedna baza prostora $V$, jer oni generišu skup $A$, pa samim tim i $V$, a nezavisni su jer su sve lin. zavisnosti vektora iz $A$ već navedene.

$\Rightarrow \dim(V)=2$

$\begin{aligned} a+3c+e&=0\\ c-d&=0\end{aligned}$

$\{a,c\}\ \checkmark \begin{array}{c} \begin{matrix} d & e \end{matrix} \\ \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \end{array} = 1$ $\quad\{c,d\}\ \times \begin{array}{c} \begin{matrix} a & e \end{matrix} \\ \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \end{array} = 0$

$\{a,d\}\ \checkmark \begin{array}{c} \begin{matrix} c & e \end{matrix} \\ \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} \end{array} = -1$ $\quad\{c,e\}\ \checkmark \begin{array}{c} \begin{matrix} a & d \end{matrix} \\ \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \end{array} = -1$

$\{a,e\}\ \checkmark \begin{array}{c} \begin{matrix} c & d \end{matrix} \\ \begin{vmatrix} 3 & 0 \\ 1 & -1 \end{vmatrix} \end{array} = -3$ $\quad\{d,e\}\ \checkmark \begin{array}{c} \begin{matrix} a & c \end{matrix} \\ \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \end{array} = 1$

(8) $v_1=(a,1,1)$

$v_2=(-a,a,-a^2)$

$v_2=(a^3,-a,1)$

$$D=\begin{vmatrix} a & -a & a^3\\ 1 & a & -a\\ 1 & -a^2 & 1\end{vmatrix} = a\begin{vmatrix} 1 & -1 & a^2\\ 1 & a & -a\\ 1 & -a^2 & 1\end{vmatrix} = a\begin{vmatrix} 1 & -1 & a^2\\ 0 & a+1 & -a-a^2\\ 0 & 1-a^2 & 1-a^2\end{vmatrix} = a\begin{vmatrix} a+1 & -a(a+1)\\ 1-a^2 & 1-a^2\end{vmatrix}$$

$$= a(a+1)(1-a^2)\begin{vmatrix} 1 & -a\\ 1 & 1\end{vmatrix} = a(1+a)^2(1-a)(1+a) = a(1+a)^3(1-a)$$

$D\neq 0 \Leftrightarrow a\in\mathbb{R}\setminus\{0,-1,1\} \Leftrightarrow$ $v_1,v_2,v_3$ su lin. nezavisni, $v_1,v_2,v_3$ generišu $V$ $\Rightarrow$ $\{v_1,v_2,v_3\}$ je baza za $V$ $\Rightarrow$ $\dim(V)=3$

• $\boxed{a=0}$

$v_1=(0,1,1)$

$v_2=(0,0,0)$

$v_3=(0,0,1)$

$\{v_1,v_3\}$ je baza $\Rightarrow \dim(V)=2$

• $\boxed{a=-1}$

$v_1=(-1,1,1)$

$v_2=(1,-1,-1)$

$v_3=(-1,1,1)$

$\{v_1\}$ je baza $\Rightarrow \dim(V)=1$

• $\boxed{a=1}$

$v_1=(1,1,1)$

$v_2=(-1,1,-1)$

$v_3=(1,-1,1)$

$\{v_1,v_2\}$ je baza $\Rightarrow \dim(V)=2$

(9) $S=\{(a,b,c,d)\in\mathbb{R}^4 \mid a+b=c+d\}$

1) $v_1=(a_1,b_1,c_1,d_1)\to a_1+b_1=c_1+d_1$

$v_2=(a_2,b_2,c_2,d_2)\to a_2+b_2=c_2+d_2$

$v_1+v_2=(a_1+a_2,\ b_1+b_2,\ c_1+c_2,\ d_1+d_2)$ $\Rightarrow v_1+v_2\in S$

$(a_1+a_2)+(b_1+b_2)=(a_1+b_1)+(a_2+b_2)$

$=(c_1+d_1)+(c_2+d_2)=(c_1+c_2)+(d_1+d_2)$

2) $v=(a,b,c,d)\to a+b=c+d$

$\alpha\in\mathbb{R}$

$\alpha v=\alpha(a,b,c,d)=(\alpha a,\alpha b,\alpha c,\alpha d)$ $\Rightarrow \alpha v\in S$

$\alpha a+\alpha b=\alpha(a+b)$

$=\alpha(c+d)=\alpha c+\alpha d$

$\Rightarrow$ $S$ jeste potprostor vektorskog prostora $(\mathbb{R}^4,\mathbb{R},+,\cdot)$

$v=(a,b,c,d)\in S \Leftrightarrow a+b=c+d$

$\Leftrightarrow a=-b+c+d$

$\Leftrightarrow v=(-b+c+d,b,c,d)$

$=(-b,b,0,0)+(c,0,c,0)+(d,0,0,d)$

$= b\underbrace{(-1,1,0,0)}_{v_1}+c\underbrace{(1,0,1,0)}_{v_2}+d\underbrace{(1,0,0,1)}_{v_3}$

$\Rightarrow$ Vektori $v_1,v_2,v_3$ generišu prostor $S$. (1) $\rightarrow$

$$A = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \rightarrow \text{rang}(A)=3 \Rightarrow \text{vektori } v_1, v_2, v_3 \text{ su lin. nezavisni } (2)$$

$(1)+(2) \Rightarrow \{v_1, v_2, v_3\}$ je baza prostora $S$

Baza za $\mathbb{R}^4$ je npr. $\mathcal{B} = \{e_1, v_1, v_2, v_3\}$

$$B = \begin{array}{c} \begin{array}{cccc} e_1 & v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \rightarrow \text{rang}(B)=4$$

(10) $\mathcal{P} = \{ p \in \mathbb{R}[x] \mid p(1)=0,\ st(p) < 4 \}$

(a) 1) $p_1, p_2 \in \mathcal{P} \Rightarrow p_1(1) = p_2(1) = 0$, $st(p_1), st(p_2) < 4$

$$\left.\begin{array}{l} (p_1+p_2)(1) = p_1(1) + p_2(1) = 0+0 = 0 \\ st(p_1+p_2) \le \max\{st(p_1), st(p_2)\} < 4 \end{array}\right\} \Rightarrow p_1 + p \in \mathcal{P}$$

2) $p \in \mathcal{P} \Rightarrow p(1)=0$, $st(p)<4$

$\alpha \in \mathbb{R}$

$$\left.\begin{array}{l} (\alpha p)(1) = \alpha p(1) = \alpha \cdot 0 = 0 \\ st(\alpha p) = st(p) < 4 \end{array}\right\} \Rightarrow \alpha p \in \mathcal{P}$$

$\Rightarrow \mathcal{P}$ je potprostor od $(\mathbb{R}[x], \mathbb{R}, +, \cdot)$

(b) $e_1(x) = x + m$

$e_2(x) = x^2 + m$

$e_3(x) = x^3 + kx^2 - lx + m$

$e_1, e_2, e_3 \in \mathcal{P} \Rightarrow e_1(1) = 0 \Leftrightarrow 1 + m = 0 \Leftrightarrow m = -1$

$e_2(1) = 0 \Leftrightarrow 1 + m = 0 \Leftrightarrow m = -1$

$e_3(1) = 0 \Leftrightarrow 1 + k + l - 1 = 0 \Leftrightarrow l = -k$

$$\alpha e_1(x) + \beta e_2(x) + \gamma e_3(x) = 0$$
$$\alpha(x-1) + \beta(x^2-1) + \gamma(x^3 + kx^2 - kx - 1) = 0$$
$$\alpha x - \alpha + \beta x^2 - \beta + \gamma x^3 + k\gamma x^2 - k\gamma x - \gamma = 0$$
$$\gamma x^3 + (\beta + k\gamma)x^2 + (\alpha - k\gamma)x - \alpha - \beta - \gamma = 0$$

$$\begin{array}{l} \gamma = 0 \\ \beta + k\gamma = 0 \\ \alpha - k\gamma = 0 \end{array} \Leftrightarrow \alpha = \beta = \gamma = 0 \Rightarrow e_1, e_2, e_3 \text{ su linearno nezavisni}$$

(c) $p(x) = x^3 + 9x^2 - 7x - 3 = (x-1)(x^2 + 10x + 3) \Rightarrow p \in \mathcal{P}$

$$\begin{aligned} p(x) = x^3 + 9x^2 - 7x - 3 &= (x^3 + x^2 - x - 1) + 8x^2 - 6x - 2 \\ &= e_3(x) + 8(x^2 - 1) - 6x + 6 \\ &= e_3(x) + 8e_2(x) - 6(x-1) \\ &= e_3(x) + 8e_2(x) - 6e_1(x) \end{aligned}$$